

# Implementing Basel II in retail banking: A simple statistical approach <sup>\*</sup>

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September 16, 2005

## Abstract

This paper provides an integrated framework for credit risk measurement and aggregation in retail banking. Tractable statistical estimation is proposed under a linear formulation with correlated residuals. Our model extends the Basel II framework, and can be accurately adapted for regulatory risk component estimation in compliance with regulatory guidelines. Practical tests on real portfolios illustrate the relevance of our general setup. The output of the estimation procedure can also be exploited for procyclicality assessment, stress testing and aggregation with corporate portfolios.

## 1 Introduction

Following the tremendous work achieved by the Basel Committee on Banking Supervision, the New Accord on Capital Adequacy has been signed in June 2004, and will take effect in 2007. Developed in BCBS (2004), it consists in a three pillar approach: minimum capital requirements, supervisory review process, and market disclosure. When compared to the 1988 accord, most innovations

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<sup>\*</sup>This article is for illustrational purpose and does not represent the official position of BNP Paribas on the Basel II issues. Figures are extracted from proprietary data but are only showing a partial view of credit risk in our bank. We would like to thank A. Chabaane, J-P. Laurent, D. Jacomy, G. Boutonnet and C. Bonnet for helpful comments and discussions. Correspondence should be addressed to A. Chouillou. The third author has received support by the Swiss National Science Foundation through the National Center of Competence: Financial Valuation and Risk Management. Part of this research was done when he was visiting THEMA and ECARES.

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appear in credit and operational risk treatments, for which quantitative methods are introduced in *Pillar I*, strong compliance requirements in *Pillar II*, and transparency with respect to the market in *Pillar III*.

Along the regulatory text BCBS (2004), quantitative methods have recently emerged. See e.g. Gordy (2000, 2002), Chabaane et al. (2004), Chouillou (2005), and Frey and McNeil (2001). Indeed the effective implementation requires a lot of work in data gathering and quantitative modelling. Numerous efforts have already been made by banks to comply with data requirements arising from BCBS (2004). Credit risk modelling in the corporate sector has also drawn a lot of attention, whereas many banks have started tackling operational risk issues. On the contrary, risk issues within retail banking can be felt as having received moderate attention from market participants and academics. Clearly there lacks a quantitative methodology designed for that particular banking activity, and fine statistical methods are strongly needed for that sector to achieve proper Basel II compliance.

The goal of this paper is to fill the gap concerning the availability of statistical tools for risk assessment under a Basel II setup in retail banking. We thus provide an integrated framework allowing for easy computations of credit risk capital requirements for retail banks. We focus on dependence structure estimation and on dealing with diversification effects within portfolios, and between portfolios of a given subsidiary and other subsidiaries of the bank. This framework will be compatible with the regulatory model, and we wish to emphasize its accuracy while keeping parsimony.

The article is organized as follows. In Section 2, we outline the general statistical model for default intensity modelling. Then we describe our general approach, before showing that the Basel II methodology can be derived as a particular case. Section 3 is devoted to empirical estimation of the key parameters on real data. They concern several portfolios coming from retail banking activities. Two versions of the model are estimated and compared. The appendix contains further details on the statistical background.

## 2 Default intensity modelling in retail banking

### 2.1 Heuristical framework

In retail banking, credits are managed on a pool basis and gathered to constitute homogeneous portfolios such as mortgages, consumer credits, loans to small and medium enterprises. These emerge from various operating and small customer oriented activities. For regulatory capital computation in the *Internal Ratings Based* (IRB) framework, borrowers in each pool have to share the same internal rating, i.e. the same risk level. Hence, in an homogeneous pool, the marginal Probability of Default (PD) and the Loss Given Default (LGD) are assumed to be the same for all borrowers for a given computational horizon.

In our framework we assume that we face a bank with  $S$  retail "subsidiaries". Each subsidiary  $s$  is made of  $J_s$  homogeneous portfolios. For each homogeneous

portfolio labelled  $j$ , we denote the aggregate exposure at the one year horizon by  $EAD_{s,j}$ . The one year marginal default probability is denoted by  $PD_{s,j}$  and the loss given default by  $LGD_{s,j}$ .

In the Basel II methodology, capital is computed from the loss distribution at the one year horizon. The modelling of the default intensity is highly emphasized whereas LGD issues are not. Even if the Committee acknowledges the importance of the latter, banks still lack accurate datasets on observed recoveries. Only rough estimates of recovery rates are available in comparison to estimates of default intensities. Thus we do not wish to incorporate LGD in the modelling at the moment.

Following Gordy (2002) and Chouillou (2005), we assume *infinite granularity* in each homogeneous portfolio. This is a working assumption which is valid in retail banking for risk indicator computation. Hence the theoretical default intensity  $pd_{t,s,j}$  in an *infinitely granular* homogeneous portfolio  $j$  in subsidiary  $s$  at time  $t$  has the following form:

$$pd_{t,s,j} = \Phi \left( \frac{\Phi^{-1}(PD_{s,j}) - \sqrt{\rho_{s,j}} \Psi_{t,s,j}}{\sqrt{1 - \rho_{s,j}}} \right), \quad (1)$$

for  $t = 1, \dots, T$ , where  $\rho_{s,j}$  is the correlation between borrowers in portfolio  $j$  of subsidiary  $s$ , and  $\Phi$  denotes the cumulative distribution function of a standard Gaussian random variable. The latent variable  $\Psi_{t,s,j}$  is an unobservable factor which follows a standard Gaussian distribution. It represents the risk in portfolio  $j$  at time  $t$ .

Formula (1) can be rewritten as:

$$\begin{aligned} \Phi^{-1}(pd_{t,s,j}) &= \frac{\Phi^{-1}(PD_{s,j,1}) - \sqrt{\rho_{s,j}} \Psi_{t,s,j}}{\sqrt{1 - \rho_{s,j}}} \\ &= \frac{\Phi^{-1}(PD_{s,j,1})}{\sqrt{1 - \rho_{s,j}}} - \sqrt{\frac{\rho_{s,j}}{1 - \rho_{s,j}}} \Psi_{t,s,j}, \end{aligned}$$

which can be further specified as a statistical linear model with correlated residuals:

$$\Phi^{-1}(pd_{t,s,j}) = \text{linear part}(t, s, j) + \text{residual}(t, s, j). \quad (2)$$

Under Formulation (2), we understand that our modelling framework is specified in terms of its linear part and the joint distribution of the residuals. In fact this type of specification is well-known and widely used in statistics. It belongs to the category of Linear Mixed Effects models.

Let us point out that a more general class of models, the so-called Generalized Linear Mixed Models, has been used by McNeil and Wendin (2003) for corporate credit risk modelling. However their framework does not apply to retail banking for which we can exploit the infinite granularity assumption in most cases.

## 2.2 Overview of Linear Mixed Effects Models

Following Laird and Waire (1982) and Demidenko (2004), we write a Linear Mixed Effects (LME) model as:

$$\mathbf{y}_t = \mathbf{X}_t\beta + \mathbf{Z}_t\mathbf{b}_t + \varepsilon_t, t = 1, \dots, T, \quad (3)$$

where:

- $\mathbf{y}_t$  is an  $n_t \times 1$  vector, with  $n_t \geq 1$ ,
- $\mathbf{X}_t$  is an  $n_t \times m$  "design" matrix of explanatory variables (fixed effects) with  $m \geq 1$ ,
- $\beta$  is an  $m \times 1$  vector of fixed effects coefficients,
- $\mathbf{Z}_t$  is an  $n_t \times k$  "design" matrix of random effects, with  $k \geq 1$ ,
- $\varepsilon_t$  is an  $n_t \times 1$  error term with independent components, each of them has zero mean and the "within" homogeneous portfolio variance  $\sigma^2$ ,
- $\mathbf{b}_t$  is a  $k \times 1$  vector of random effects with zero mean and covariance matrix  $\mathbf{D}^*$ .

All random vectors  $\{\mathbf{b}_t, \varepsilon_t; t = 1, \dots, T\}$  are mutually independent and normally distributed:

$$\varepsilon_t \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_{n_t}), \mathbf{b}_t \sim \mathcal{N}(0, \mathbf{D}^*),$$

but we do not know the variance  $\sigma^2$ , nor the matrix  $\mathbf{D}^*$ . Besides, we have assumed that there are  $m$  explanatory variables,  $m \geq 1$ , for the fixed effects. Typically we may include the subsidiary, the country, and an internal rating in the explanatory variables  $\mathbf{X}_t$ . We may further add the sector and other possible features such as time-to-maturity.

## 2.3 From Basel II to a more general correlation structure

### 2.3.1 Basel II estimation

The standard methodology in Basel II imposes  $\mathbf{D}^* = 0$ . Indeed the regulatory requirement of independent estimation across homogeneous portfolios implicitly means independence of factors. In that case we may simply use an Ordinary Least Squares approach to estimate  $\beta$ :

$$\hat{\beta}_{OLS} = \left( \sum_t \mathbf{X}_t' \mathbf{X}_t \right)^{-1} \left( \sum_t \mathbf{X}_t' \mathbf{y}_t \right). \quad (4)$$

We can construct confidence intervals and various statistical tests for parameter significance. According to an observation from Demidenko (2004), this constrained estimate of  $\beta$  is often numerically close to the unconstrained one under

the general setup (3). However confidence intervals are usually wider and frequently yield rejection of the null hypothesis. This means that random mixed effects have to be statistically taken into account in most common practical situations.

### 2.3.2 A model with compound symmetry

We first estimate a model in which there is no differentiation of subsidiaries in the correlation structure, though we differentiate them in the linear part of the model. This model follows the framework described by Chabaane et al. (2004), and corresponds to compound symmetry in the correlation structure of  $\mathbf{y}_t$ :

$$\mathbf{y}_t = \mathbf{X}_t\beta + b_t + \boldsymbol{\varepsilon}_t, \quad (5)$$

with

$$b_t \sim \mathcal{N}(0, \sigma_{sys}^2), \boldsymbol{\varepsilon}_t \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_{n_t}).$$

This yields the following distribution for the vector  $\mathbf{y}_t$ :

$$\mathbf{y}_t \sim \mathcal{N}(\mathbf{X}_t\beta, \boldsymbol{\Omega}_t),$$

with

$$\boldsymbol{\Omega}_t = cov(\mathbf{y}_t) = \begin{bmatrix} \sigma_{sys}^2 + \sigma^2 & \sigma_{sys}^2 & \sigma_{sys}^2 \\ \sigma_{sys}^2 & \cdot & \sigma_{sys}^2 \\ \sigma_{sys}^2 & \sigma_{sys}^2 & \sigma_{sys}^2 + \sigma^2 \end{bmatrix}.$$

A model with compound symmetry is constrained in allowing only for one systemic risk in the whole economy. Model (5) corresponds indeed to a parsimonious model with  $k = 1$ .

In this setup, we define the systemic correlation  $\rho_{sys}$  and the specific correlation  $\rho$  as:

$$\rho_{sys} = \frac{\sigma_{sys}^2}{\sigma^2 + \sigma_{sys}^2}, \quad (6)$$

$$\rho = \frac{\sigma^2 + \sigma_{sys}^2}{1 + \sigma^2 + \sigma_{sys}^2}. \quad (7)$$

Let us remark that the specific correlation (7) corresponds to the Basel II correlation in BCBS (2004), whereas the systemic correlation (8) is assumed to be 0% in the Basel II estimation process and 100% in the regulatory capital computation process. On one hand it is 0% in the estimation process as the regulatory framework asks for stand-alone estimation of the risk components of each portfolio. On the other hand it is 100% for capital computation as capital is additive in the IRB approach. Such a methodological inconsistency do exist in the Basel II guidelines, and we should be aware of it.

### 2.3.3 A finer correlation structure for subsidiary aggregation

As before we consider that there are  $J_s$  homogeneous portfolios in each subsidiary  $s$ . For  $s = 1, \dots, S$ ,  $j = 1, \dots, J_s$ , and  $t = 1, \dots, T$ , we observe the default intensity  $pd_{t,s,j}$  of the homogeneous portfolio  $j$  in subsidiary  $s$  at date  $t$ , which can be transformed into:

$$y_{t,s,j} = \Phi^{-1}(pd_{t,s,j}).$$

Using a vector notation, we can write:

$$\mathbf{y} = \left( \left( \left( \begin{pmatrix} y_{1,1,1} \\ \vdots \\ y_{1,1,J_1} \end{pmatrix} \right)', \dots, \left( \begin{pmatrix} y_{1,S,1} \\ \vdots \\ y_{1,1,J_S} \end{pmatrix} \right)' \right), \dots, \left( \dots, \left( \begin{pmatrix} y_{T,S,1} \\ \vdots \\ y_{T,1,J_S} \end{pmatrix} \right)' \right) \right)'.$$

The specification of the "design" matrix  $\mathbf{Z}_t$  is block diagonal:

$$\mathbf{Z}_t = \begin{bmatrix} \mathbf{1}_{J_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{J_S} \end{bmatrix},$$

where  $\mathbf{1}_{J_s}$  denotes a unit column vector of size  $J_s$ . It will correspond to the "systemic" risk structure of each subsidiary  $s$ . All the  $\mathbf{Z}_t$  are the same here as we follow the same cohorts from  $t = 1$  up to  $t = T$ . This is not a necessary condition of the modelling framework. However, there must be at least one portfolio for each subsidiary from the beginning, otherwise we would have an unidentified statistical model.

The vector  $\mathbf{b}_t$  of random effects is of size  $k = S$ . There is one systemic risk by subsidiary, and they are correlated through  $\mathbf{D}^* = cov(\mathbf{b}_t)$ . This matrix does not depend on time. For computation purpose, we opt for a parametric correlation structure for the subsidiary risks:

$$cov(\mathbf{b}_t) = \begin{bmatrix} \sigma_r^2 + \sigma_{sub}^2 & \sigma_r^2 & \sigma_r^2 \\ \sigma_r^2 & \ddots & \sigma_r^2 \\ \sigma_r^2 & \sigma_r^2 & \sigma_r^2 + \sigma_{sub}^2 \end{bmatrix},$$

with  $\sigma_r^2$  and  $\sigma_{sub}^2 > 0$ . In this setup, we rely implicitly on one risk factor common to all subsidiaries, and one risk factor specific to each subsidiary. If needed we may further refine the model by introducing country risk differentiation within each subsidiary. This will just increase the notational burden.

The error term  $\varepsilon_t$  is defined as in the general LME model, and corresponds to independent components, which are normally distributed with zero mean and variance  $\sigma^2$ .

Hence we arrive at the same formulation as in (3):

$$\mathbf{y}_t = \mathbf{X}_t\beta + \mathbf{Z}_t\mathbf{b}_t + \varepsilon_t. \quad (8)$$

This means that we obtain a simple and straightforward statistical framework ready for subsidiary aggregation. It first takes into account specific risk within each homogeneous pool, then systemic risk within a subsidiary, and finally systemic risk between subsidiaries. All the necessary and sufficient conditions for ensuring the existence of the Maximum Likelihood Estimate and its asymptotic properties are given in Appendix A and developed at length in Demidenko (2004).

Let us remark that the "retail" correlation denoted here by  $\rho_r$  can be defined as:

$$\rho_r = \frac{\sigma_r^2}{\sigma_r^2 + \sigma_{sub}^2}.$$

This correlation gives an indicator of how close the subsidiaries are in terms of credit risk comovements. Furthermore, the reader can note that we can define a correlation  $\rho_{sub}$  between portfolios, within a subsidiary:

$$\rho_{sub} = \frac{\sigma_r^2 + \sigma_{sub}^2}{\sigma_r^2 + \sigma_{sub}^2 + \sigma^2},$$

whereas the "specific" correlation  $\rho_s$  within a portfolio, assumed to be the same for all portfolios and all subsidiaries, has the following form:

$$\rho_s = \frac{\sigma_r^2 + \sigma_{sub}^2 + \sigma^2}{1 + \sigma_r^2 + \sigma_{sub}^2 + \sigma^2}.$$

### 3 Empirical estimation on the whole retail perimeter

#### 3.1 Setup

##### 3.1.1 Input data

Panel data from the retail subsidiaries are the necessary input to the estimation process. For this study, we have extracted proprietary data from two subsidiaries of BNP Paribas. Those data display yearly default rates on homogeneous credit portfolios segmented according to:

- the subsidiary: two subsidiaries operating in two distinct segments of the mortgage market.
- the internal rating: four risk levels, 1 to 4, from the lowest to the highest creditworthiness. The rating is assigned to a credit at inception of the credit, i.e. there is no dynamics of ratings in our setup.
- the vintages: 19 generations, from 1985 to 2003.

There are 19 years of observations, from  $t = 1985$  to  $T = 2003$ . As time passes new homogeneous portfolios appear according to our segmenting rule.

Note that LME models are particularly effective in studying that kind of data for which the number of observations may vary from time to time. We have a longer history for older credits, and a shorter history for more recent credits. An LME approach allows us to fit parameters by taking advantage of the larger information available for the older credits.

### 3.1.2 Estimation software

We have estimated the model with the free software R, downloadable at [www.r-project.org](http://www.r-project.org). In the "nlme" package, the "lme" function implements log-likelihood maximization, and various functions are available to assess the statistical validity of the model. An estimation package is also available in SAS and S-plus. Direct implementation can be tricky as technical problems may arise (size of matrices and optimization procedures), but suitable routines in C++ are available (<ftp://ftp.stat.wisc.edu/src/NLME>).

### 3.1.3 Specifying the linear part of the model

We have decided to give a special form to the linear part of the model. Thus, we will regress on the dummy "internal rating" coupled with the vintage. Next we add dummy variables to account for seasoning effects.

### 3.1.4 Fitted results: the linear part of the model

**Model with compound symmetry** The fixed effects parameters have been estimated, and are highly significant at the 95% confidence level. The figure below displays the transformed default intensity plotted against the estimated ones, from time to time. Most points are found for recent times, as new pools are created each year. One can inspect visually the quality of the fit on Figure 1, in which time has been translated to 0 (corresponding to 1985) up to 19 (corresponding to 2003).

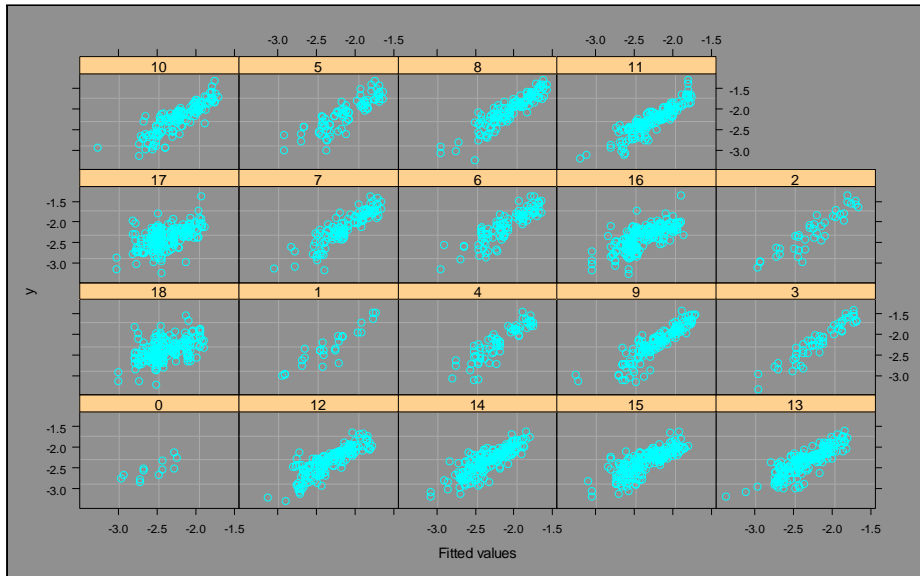


Figure 1: Observed versus fitted values grouped by date in the model with compound symmetry

**General model** We now estimate the general model. The linear part is specified in the same way as for the model with compound symmetry. In our particular case, there seems to be little improvement by switching to the general model (8). This will be confirmed by the estimation of the correlation structure in the two models.

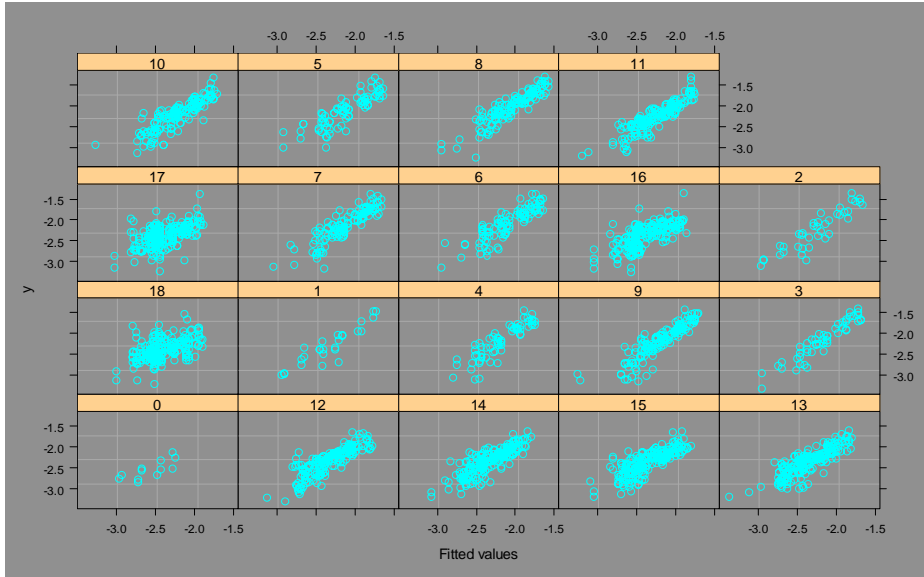


Figure 2: Observed versus fitted values grouped by date in the general model

### 3.1.5 Estimation of the variance of random effects and residuals

**Model with compound symmetry** The estimation of random effects obtained by the lme function of R are displayed below.

	est.
$\sigma_{sys}$	0.05243387
$\sigma$	0.2041051

A check at the confidence intervals is easily obtainable from R:

	lower	est.	upper
$\sigma_{sys}$	0.03557078	0.05243387	0.07729126
$\sigma$	0.1980193	0.2041051	0.2103778

Finally, this yields our estimation of the correlation structure between portfolios and within portfolios:

$\rho_{sys}$	6.19%	(9)
$\rho$	4.26%	

**General model** We see that the correlation between portfolios is at a very low level, which partly validates the Basel II methodology of performing separate estimation on each homogeneous portfolios. However from a statistical point of view the correlation is significantly different from zero. Besides, the within portfolio correlation is not very high, which indicates that Basel II has put an emphasis on conservative assumptions.

$\sigma_r$	0.04839159
$\sigma_s$	0.02679534
$\sigma$	0.2042083

The estimated values are:

	lower	est.	upper
$\sigma_r$	0.03047506	0.04839159	0.07684137
$\sigma_s$	0.01319025	0.02679534	0.05443341
$\sigma$	0.1980934	0.2042083	0.210512

This yields the following correlation structure:

$\rho_r$	76.53%	(10)
$\rho_s$	6.84%	
$\rho$	4.28%	

The retail correlation  $\rho_r$  is indeed very high, whereas the other correlations are low. For capital computation purpose, this is in favour of adding up the capital of the first subsidiary to the capital of the second subsidiary.

Besides, we can see that estimation of model (8) gives very close results to the previous estimation in the more general model (3) setup. Indeed, model (5) yields in (9) a systemic correlation of 6.19% to be compared with  $\rho_s = 6.84\%$ , whereas the within portfolio correlation  $\rho = 4.26\%$  is roughly equal to 4.28%. Indeed the correlation  $\rho_r$  is equal to 76.53% whereas model (5) assumes it to be equal to 100% implicitly. Overall we can say that model (8) does not really bring further explanatory power compared to model (5) for this particular empirical application, albeit some features are statistically significant.

### 3.1.6 Further estimations and applications

**Random effects, stress testing and procyclicality** Following Demidenko

(2004), we can estimate the mixed random effects conditional on the default intensities. More precisely, we compute  $\mathbb{E}(\mathbf{b}_t | \mathbf{y}_t)$  under the Gaussian assumption, for  $t = 1, \dots, T$ . Thus, we obtain a time-series  $\hat{\mathbf{b}}_t$  which can then be regressed over a set of observable variables, such as the Consumer Price Index, the unemployment rate, the savings rate, or other key variables for explanatory

purposes. This should yield a better understanding of retail banking risk factors, and should ease strategic management decisions in that sector. Moreover, any procyclicality effect could be dampened by a forward looking assessment of future systemic risk. See Gordy (2004). Thus we can create scenarios for stress test and assessment of future capital needs should the economy improve or deteriorate.

**Estimation of random effects in model (5)** Figure 3 plots the estimates of the random effect part in the model with compound symmetry.

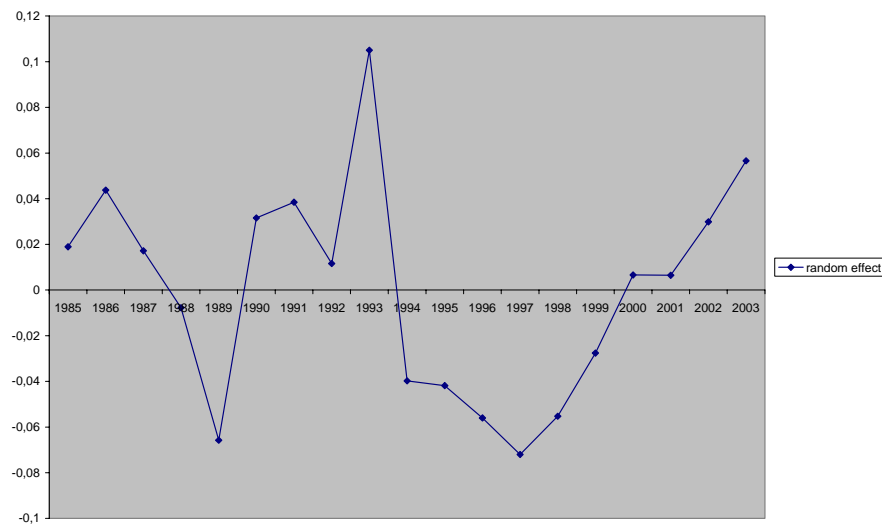


Figure 3: Time series of random effects in the model with compound symmetry

**Estimation of random effects in model (8)** It is possible to do the same estimation in the case of model (8) where there is one systemic risk by subsidiaries. This yields two time series displayed in Figure 4. Such an analysis allows us to disentangle risk stemming from each subsidiary. Hence this can yield a fine use of stress testing.

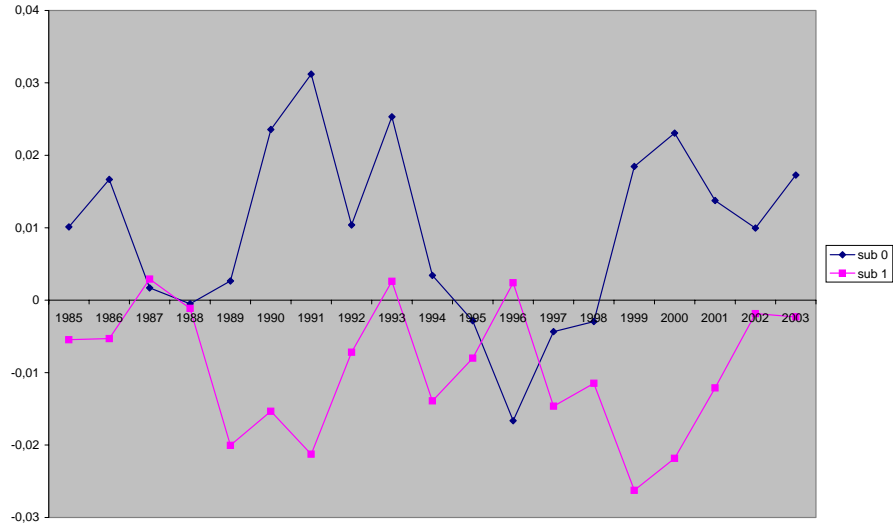


Figure 4: Time series of random effects in each subsidiary in the general model

### 3.1.7 Validation of the rating methodology

Rating are assigned to credits in order to pool them through a ranking of their riskiness. Our estimation can validate the quality of the rating methodology as a way to discriminate risks: the vintage being fixed, we can follow the default rates as a function of the age. The higher the curve, the riskier the credit portfolio on a marginal basis.

Here we can see ex post that the rating methodology is coherent. Moreover, we can observe the peak in default after origination, thus bad credits leave the pool, followed by a decrease in realized default rates. This kind of humped shape has been well documented in the literature on credit risk under the so-called "seasoning" effect. This "seasoning" behavior is akin to the one observed on prepayment rates in mortgage-back-securities (MBS).

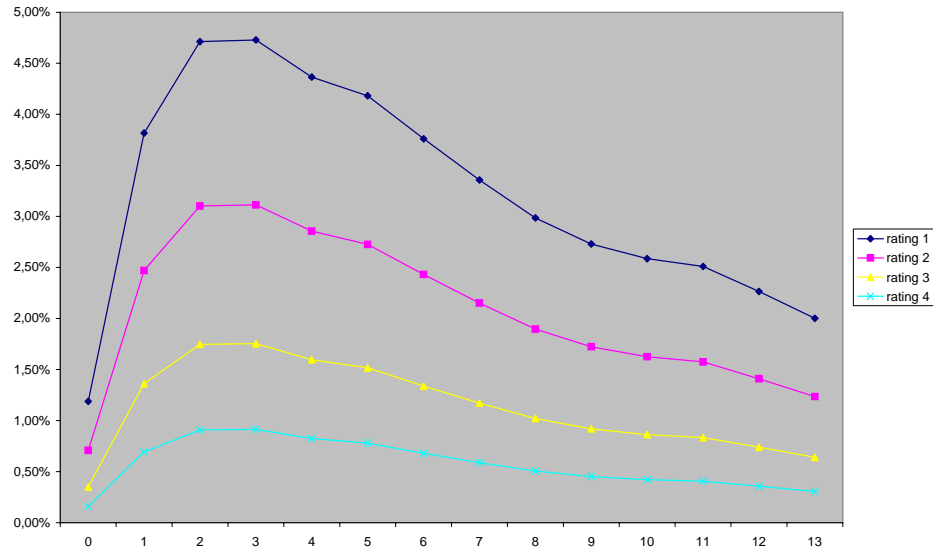


Figure 5: Ranking of riskiness and seasoning effect

## 4 Conclusion and applications

This article suggests a tractable framework which allows for joint estimation of marginal default probability and correlation across all the retail portfolio within a rigorous statistical approach. Depending on the user specified correlation structure, we can obtain either a Basel II estimation setup or a more general setup. Requirements for the estimation are infinite granularity, the availability of a scoring method, and a segmentation of the portfolio by sector, products and vintage.

The empirical results have shown the relevance of the statistical model and shown its use both to assess default rates and a forward looking view on capital requirements and systemic risk. The linear part of the model allows the risk manager to incorporate the desired effects, and test their validity. This is very useful to determine important covariates for stress testing methods.

Armed with this model, we can also compute capital, and investigate the impact on alternative risk measures. Extensive empirical studies have already been made in Chabaane et al. (2004) and Chouillou (2005). Finally, our statistical framework can be coupled with the corporate sector in order to allow a regulatory and economic capital computation at the bank level.

## A Estimation of Linear Mixed Effects models

### A.1 Log-likelihood function

Let us introduce  $N_T = \sum_{t=1}^T n_t$  and the *vech* operator, following Magnus (1988), which transforms a symmetric  $k \times k$  matrix  $\mathbf{M}$  into a  $k(k+1)/2 \times 1$  vector made of the elements of its upper diagonal part. This means that *vech* ( $\mathbf{M}$ ) contains the distinct elements of the matrix ( $\mathbf{M}$ ).

For model (3), it is better to introduce matrix  $\mathbf{D}$  which is such as  $\mathbf{D}^* = \sigma^2 \mathbf{D}$ . This will ease the expression of the log-likelihood function.

Dropping the constant  $-(N_T/2) \ln(2\pi)$ , the log-likelihood function for the LME model (3) is given by:

$$l(\theta) = -\frac{1}{2} \left\{ N_T \ln \sigma^2 + \sum_{t=1}^T \left[ \ln |\mathbf{I}_{n_t} + \mathbf{Z}_t \mathbf{D} \mathbf{Z}_t'| + \sigma^{-2} \mathbf{e}_t' (\mathbf{I}_{n_t} + \mathbf{Z}_t \mathbf{D} \mathbf{Z}_t')^{-1} \mathbf{e}_t \right] \right\}, \quad (11)$$

where

$$\mathbf{e}_t = \mathbf{y}_t - \mathbf{X}_t \beta$$

and

$$\theta = (\beta, \sigma^2, \text{vech}(\mathbf{D}))$$

is a vector of unknown parameters. Hence the total dimension of the parameter vector  $\theta$  is  $m + 1 + k(k+1)/2$ . The Maximum Likelihood Estimate (MLE) maximizes the log-likelihood function  $l$  over the parameter space

$$\Theta = \{ \theta : \beta \in \mathbb{R}^m, \sigma^2 > 0, \mathbf{D} \text{ nonnegative definite} \}.$$

The asymptotic covariance matrix of the estimators can be found in Demidenko (2004), and is given as the inverse of the information matrix.

### A.2 Condition for identifiability

Folowing Demidenko (2004), the following condition is necessary and sufficient for ensuring identifiability of the model: if matrix  $\mathbf{X}$  has full rank, at least one matrix  $\mathbf{Z}_t$  has full rank and  $\sum (n_t - k) > 0$ , the LME model is identifiable.

### A.3 Condition for the existence of the Maximum Likelihood Estimate

On the conditions for the existence of the Maximum Likelihood Estimate, see Rao and Kleffé (1988), Demidenko and Massam (1999) and Demidenko (2004). A theorem by Demidenko (2004) provides necessary and sufficient conditions for the existence of the MLE: in the LME model, the MLE exists with probability 1 if the total number of observations is "sufficiently" large, in particular if  $\sum (n_t - k) - m > 0$ . For further details, see Hartley and Rao (1967), Harville

(1977), Henderson (1953), Henderson (1984), Henderson (1990) and Wolfinger et al. (1994).

#### A.4 Maximization algorithms

There are three kinds of algorithms for maximizing the log-likelihood function (5):

- Expectation-Maximization,
- Fisher Scoring,
- Newton-Raphson.

All these algorithms can be applied in the framework of the Linear Mixed Effects model, Harville (1974), Jennrich and Schluchter (1986), Laird, Lange et al. (1987), Lindstrom and Bates (1988), Jamshidian and Jennrich (1993), Wolfinger et al. (1994), Demidenko (2004).

The initialization of the algorithm can be done with a non iterative distribution free quadratic estimation of  $\mathbf{D}$  and  $\sigma^2$ . The fixed effects vector  $\beta$  is then initialized through a Generalized Least Square regression. Demidenko (2004) provides three quadratic estimators which have good properties: consistency and unbiasedness, and which satisfy optimality criteria. Moreover, those estimators are equal to the MLE in some cases, hence the algorithm converges at the first iteration in such cases.

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