Collateralized Debt Obligations pricing and factor models: a new methodology using Normal Inverse Gaussian distributions.

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Abstract

The reported "correlation smile" in the CDO market is proof that the spreads of CDOs tranches are not consistent when we use the widely-known Gaussian one-factor model for the pricing. We introduce a new methodology in which non-standard tranches such as bespoke single tranches can be valued. The underlying idea of our framework is to use the tranches' price quotes available in the market to determine the implied distribution of the common factor for a given correlation level. In our methodology the estimated correlation between the underlying assets of a CDO's underlying portfolio becomes an input. We propose an improvement to the market standard model by using Normal Inverse Gaussian distributions and we show that our approach is theoretically and empirically more accurate.

JEL classification: G12, G13.

Keywords: CDO pricing, implied correlation, implied distribution, loss distribution, factor model, Normal Inverse Gaussian distribution, default probability, conditional default probability.
1 Introduction

Synthetic collateralized debt obligations (CDOs) have been the principal growth engine for the credit derivatives market over the last few years. They create new, customized asset classes by allowing various investors to share the risk and return of an underlying portfolio of credit default swaps\(^1\) (CDS). Multiple tranches of securities are issued by the CDO, offering investors various maturity and credit risk characteristics. Thus, the attractiveness to investors is determined by the underlying portfolio of CDS and the rules for sharing the risk and return.

A synthetic CDO is often called "a correlation product" because, in simple words, it is a contract that references the default of more than one obligor. Investors in this product are buying correlation risk, or more exactly, joint default risk between several obligors. The underlying portfolio loss distribution directly determines the tranche cash flows and thus the tranche valuation. The Gaussian one-factor model has become the established way of pricing correlation products.

In 2003, the iBoxx and Trac-x\(^2\) portfolios were introduced, and tranches linked to these reference sets also started to be actively quoted. This portfolio standardization has allowed for the creation of a more liquid and transparent market for CDO tranches. The new availability of relatively liquid market levels has led to the price quotes of the tranches in terms of implied compound correlation.

The Gaussian one-factor model does not provide an adequate solution for pricing simultaneously various tranches of an index, nor for adjusting correlation against the level of market spreads. Thus, the implied compound correlation is the uniform asset correlation number that makes the fair or theoretical value of a tranche equal to its market quote. There is a reported "correlation smile" in the CDO market. Friend and Rogge (2004), Green-

\(^1\)In its basic form, a credit default swap (CDS) is essentially a contract that transfers default risk from one party to another; the risk protection buyer pays the protection seller a premium (spread), usually in the form of a semi-annual annuity.

\(^2\)CDS indices. Nowadays, the iTraxx Europe which is product of a merger between iBoxx and Trac-x, is the most popular CDS index in Europe. The CDX is the North American CDS index.
berg et al (2004), Finger (2005) also report such an effect meaning that the Gaussian one-factor model fails to price accurately the observed prices of CDS index tranches. Thus, we need to develop a coherent framework in which either index tranches or non-standard CDOs tranches, such as bespoke\textsuperscript{3} single tranches, could be valued.

There has been much interest recently in simple extensions of the Gaussian one-factor model in order to match the "correlation smile" in the CDO market. Andersen, Sidenius and Basu (2003) suggest a principal components analysis to build a low dimensional correlation structure from the correlation matrix based on the firms' equity returns. Gregory and Laurent (2004) propose a correlation structure built from groups specifying intra and inter-group correlation coefficients and they introduce some dependence between recovery rates and defaults. Hull and White (2005) recommend the use of a double Student-t one-factor model. Andersen and Sidenius (2005) introduce random recovery rates and random factor loadings in the model. Burtschell, Gregory and Laurent (2005) propose a comparative analysis of the previous CDO pricing models and illustrate the fact that these models should be improved.

The underlying idea of our framework is to use the price quotes of the tranches available in the market to determine the implied distribution of the common factor for a given input correlation level. In the economic theory, the common factor of the model reflects the general state of the business cycle. In practice, we can assume that this general state is reflected by the returns of equity or bond indices. However, it is now a well-known fact that returns from financial market variables are characterized by non-normality. The empirical distribution of such returns is more peaked and has fatter tails than the Gaussian distribution, which implies that very large changes in returns occur with a higher frequency than under normality. In addition, it is often skewed, so the use of symmetric distributions as Student’s t-distributions is restrictive. A promising distribution for such returns proposed in the literature is, in particular, the Normal Inverse Gaussian distribution (NIG).

In order to take into account the previous remarks, in this paper we introduce NIG-distributions in the CDO pricing framework. This distribution

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\textsuperscript{3}The word bespoke means "custom-made".
has the remarkable property of being able to represent stochastic phenomena that have heavy tails and/or are skewed. We study the sensitivity of the tranches’ prices to the Kurtosis and Skewness of the distributions used in the factor model. Then, we show that the use of NIG-distributions improves the model and matches market data by correcting the ”correlation smile”.

2 Synthetic CDO structure and implied correlation

In some CDOs the underlying portfolio is composed of CDS rather than bonds or loans. Because CDS permit ”synthetic” exposure to credit risk, a CDO backed by CDS is called a synthetic CDO. By contrast, a CDO backed by ordinary bonds or loans is called a cash CDO. Synthetic CDOs recently have become very popular, especially in Europe.

A synthetic CDO receives periodic fees (spread) as a protection seller. The periodic fees provide the source of funds for the CDO to pay a premium (spread) to investors who hold the tranches issued by the CDO. Each tranche is defined by an attachment and an exhaustion point. Tranches are categorized as super senior, senior, senior mezzanine, junior mezzanine and subordinated/equity, according to their degrees of credit risk. Payments to senior tranches take precedence over those of mezzanine tranches, and payments to mezzanine tranches take precedence over those to subordinated/equity tranches. If a credit event occurs under any CDS in the underlying portfolio, the CDO would be required to pay the protection buyer under the CDS. The CDO would first use some of the money invested by the equity holders. Thus, the CDO’s assets would decline and it might not be able to fully
repay its outstanding issued tranches. For investors in the synthetic CDO, the occurrence of a credit event under any CDS in the underlying portfolio has essentially the same effect as if the CDO had purchased a bond that subsequently defaulted. The main risk for a synthetic CDO is through its underlying portfolio of CDS.

Beyond synthetic CDOs, there are "single-tranche CDOs". The underlying structure of a single-tranche synthetic CDO is very similar to that of more traditional, multiple-tranche synthetics. As in a full-structure synthetic CDO, credit risk is transferred through a portfolio of CDS. The main difference is that, in a single-tranche transaction, only a specific portion of the portfolio's risk - rather than the entire capital structure - is transferred to the investor. A single-tranche transaction is sometimes referred to as bespoke, because the investor can customize various characteristics such as the portfolio composition, maturity, credit rating, tranche size and subordination, management/substitution rights, issued currency, etc.

![Figure 1: Standard synthetic CDO structure in Europe.](image-url)
Figure 1 illustrates the standard CDO structure in Europe. The tranches presented in this figure are the ones quoted by dealers on a daily basis in the Euro market. Currently, in Europe, tranches of the iTraxx\textsuperscript{4} Europe Series 3 index constitute the set of liquid instruments, while in the US tranches of the CDX.NA.IG index play the same role. The attachment and exhaustion points of the standard tranches evolved to create instruments with distinct risk profiles. The first-loss 0-3% equity tranche is exposed to the first several defaults in the underlying portfolio, and offers high returns if no defaults occur. The 3-6% and the 6-9% tranches, the junior and senior mezzanine, are levered in the underlying portfolio spread, but are less immediately exposed to the portfolio defaults. The 9-12% tranche is the senior tranche, while the 12-22% tranche is the low-risk super senior piece.

In table 1 we illustrate\textsuperscript{5} the price quotes of the iTraxx Europe Series 3 on the 28\textsuperscript{th} of April 2005 for three different maturities (5, 7, and 10 years). The spread of a tranche is the premium (in basis points) over the Euribor\textsuperscript{6} that is paid to investors on a yearly basis. This spread is determined by bid and offer on the market. In practice, only equity tranches are traded with an upfront premium, as do distressed credits in the CDS market. To link spread and correlation, the market standard model is the one-factor Gaussian model. The implied compound correlation (IC correlation) is the uniform asset correlation number that makes the fair or theoretical value of a tranche equal to its market quote.

\textsuperscript{4}The iTraxx Europe Series 3 is composed of 125 CDS.
\textsuperscript{5}The WAS is the Weighted Average Spread of the index.
\textsuperscript{6}Euribor stands for Euro interbank offered rate. These interest rates for the Euro are compiled by the European Banking Federation. Euribor is widely used as the underlying risk-free rate for Euro-denominated derivative contracts.
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Table 1: iTraxx Europe Series 3 / Valuation date: 28-Apr-05.

The tranches’ implied compound correlations are not the same across the structure. If we plot the implied compound correlation for each tranche and different maturities as in figure 2, we get three different kinds of ”smile”\(^7\).

![Figure 2: Implied compound correlation (%) by tranche for different maturities.](image)

\(^7\)The analysis is directly extendable to the CDX.NA.IG index in the US.
The general “correlation smile” bears many similarities to the more familiar implied volatility “smile” from the options’ market. First, it is a mean of quoting the spread on a tranche. Second, and more importantly, the “smile” reflects the market implied loss distribution of the underlying index. A possible justification for the “smile” is that the market ascribes a higher probability for extreme loss scenarios, i.e., the tails of the loss distribution in the underlying portfolio are fatter than those described by the normal distribution assumptions of the Gaussian one-factor model.

We illustrate in figure 3 that the correlation parameter plays a key role in determining the shape of the portfolio loss distribution. The higher the correlation parameter value, the fatter are the tails of the distribution. Fatter tails mean that extreme loss outcomes (both very low levels of defaults and very high levels) are more likely relative to average defaults level. Correspondingly, a low correlation results in a portfolio loss distribution with skinny tails, meaning that the average loss outcome is more likely relative to extreme default outcomes.

Figure 3: Impact of correlation on the portfolio loss distribution.

Having introduced the synthetic CDO structure and addressed the ”correlation smile”, we now move on to presenting the general properties of factor models.
3 Factor Models

In this section we present a general definition of the factor models that are used in the CDO market and that we are going to extend in section 5 using NIG distributions. Factor models represent a useful and efficient framework to model the dependency structure for portfolios underlying most synthetic CDOs. They are also used to derive portfolio loss distributions.

The idea behind factor models\(^8\) is to break down the firms’ asset values into a risk component that is idiosyncratic to the asset, plus one or a number of factors that are systematic to all assets in the portfolio. The obligors of the CDO’s underlying portfolio are defined as firms and we consider the firms’ asset values as a function of a group of common factors, which introduce the default correlation in the model, plus a firm’s specific factor. This methodology is useful in order to combine the information content for several different variables into one (or a few) representative factor. We also assume that a firm defaults when its asset value falls below a certain default barrier. This computation corresponds to its conditional default probability.

To obtain the underlying portfolio loss distribution of a CDO using a factor model we need to proceed in two steps. In the first step, the aggregation step, we model the risk behavior and joint default correlation between firms by calculating their default probabilities conditionally to common factors. Under such modelling assumptions, it is possible to numerically compute the Fourier transform of the underlying portfolio’s aggregate loss distribution. Thus, in the second step called the inversion step, we obtain the underlying portfolio loss distribution and then the spread of each tranche by numerically computing the inverse Fast Fourier Transform\(^9\) or some other inversion method. Because the second step is an analytical procedure well described in the literature and also because the behavior of the underlying portfolio loss distribution entirely depends on the way by which we construct the aggregation step, in this section we only focus on the first step.

\(^8\)The use of factor models in credit risk management is reportedly due to Vasicek (1997). This approach is also used in Belkin et al. (1998), Finger (1999), Schonbucher, and Frey, McNeil and Nyfeler (2001). The pricing of CDOs using factor models has been also studied by Andersen, Sidenius and Basu (2003), Laurent and Gregory (2003) and similar techniques were later proposed by Hull and White (2004).

For more legibility, we first introduce a one-factor model with a common factor $Z$ and then we generalize our approach with an $m$-factor model.

For $i = 1, ..., n (n \in \mathbb{N})$, we define $V_i$ the $i^{th}$ firm’s asset value as:

$$V_i = \rho_i Z + \omega_i \varepsilon_i,$$  \hspace{1cm} (1)

where:

- $Z$ is the common factor of the model,
- $\varepsilon_i$ is the idiosyncratic risk of the $i^{th}$ firm,
- $V_i$, $Z$ and $\varepsilon_i$ are independent random variables with zero-mean and unit-variance distributions,
- $\rho_i$ represents the sensitivity of $V_i$ to $Z$ with respect to $-1 \leq \rho_i \leq 1$,
- $\omega_i$ represents the sensitivity of $V_i$ to $\varepsilon_i$.

Thus, the expectation and the variance of $V_i$ are given by:

$$E[V_i] = \rho_i E[Z] + \omega_i E[\varepsilon_i] = 0,$$ \hspace{1cm} (2)

and,

$$Var[V_i] = \rho_i^2 Var[Z] + \omega_i^2 Var[\varepsilon_i] + 2\rho_i \omega_i Cov[Z, \varepsilon_i].$$ \hspace{1cm} (3)

Thus,

$$Var[V_i] = \rho_i^2 + \omega_i^2.$$ \hspace{1cm} (4)

Then, there exists relationship between $\rho_i$ and $\omega_i$:

$$\omega_i = \sqrt{1 - \rho_i^2},$$ \hspace{1cm} (5)

and the equation (1) becomes:

$$V_i = \rho_i Z + \varepsilon_i \sqrt{1 - \rho_i^2}.$$ \hspace{1cm} (6)
The correlation between the two random variables $V_f$ and $V_j$ defined previously is given, for $1 \leq f \leq n$ and $1 \leq j \leq n$, by:

$$Corr[V_f, V_j] = \frac{Cov[V_f, V_j]}{\sqrt{Var[V_f]} \sqrt{Var[V_j]}},$$

(7)

that reduces to:

$$Corr[V_f, V_j] = E[V_f V_j].$$

(8)

Now using equations (8) and (6), we get the following relationship:

$$Corr[V_f, V_j] = \rho_f \rho_j E[Z^2] = \rho_f \rho_j.$$

(9)

In this part we study the conditional default probability of the $i^{th}$ firm. For that we assume that a firm $i$ defaults when its asset value hits the barrier $k_i$ ($k_i \in R$). Thus, looking at the expression of the $i^{th}$ firm's asset value given in equation (6), we argue that for specific realizations of the common factor $Z$, the $i^{th}$ firm defaults as soon as its asset value hits the default barrier $k_i$. We denote $P_i(Z)$ the conditional default probability of the $i^{th}$ firm, then for $Z = z$:

$$P_i(z) = \text{Prob}[V_i \leq k_i | Z = z].$$

(10)

Using equation (6), the expression (10) becomes:

$$P_i(z) = \text{Prob} \left[ \varepsilon_i \leq \frac{k_i - \rho_i z}{\sqrt{1 - \rho_i^2}} | Z = z \right].$$

(11)

We denote $C_{\varepsilon_i}$ the cumulative distribution for the random variable $\varepsilon_i$, then the probability given in (11) for $Z = z$ and for $i = 1, ..., n$ is equal to:

$$P_i(z) = C_{\varepsilon_i} \left[ \frac{k_i - \rho_i z}{\sqrt{1 - \rho_i^2}} \right].$$

(12)

In order to determine the real value of the default barrier $k_i$ for each firm $i$, we denote $Q_i$ the unconditional default probability of the $i^{th}$ firm. $Q_i$ is recovered from CDS market data using an intensity-based model\(^{10}\). By

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\(^{10}\)The first published intensity model appears to be Jarrow and Turnbull (1995). Subsequent research includes Duffie and Huang (1996), Jarrow, Lando and Turnbull (1997) and Duffie, Singleton (1997a, 1997b) and Schonbucher and Schubert (2001). The fundamental idea of the intensity-based model framework is to model the default probability as the first jump of a Poisson process.
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definition we have:

\[ Q_i = \text{Prob}[V_i \leq k_i] = E[P_i(Z)]. \] (13)

If \( \phi \) represents the probability distribution for the random variable \( Z \), then:

\[ E[P_i(Z)] = \int_{-\infty}^{+\infty} P_i(z)\phi(z)dz, \] (14)

which implies that:

\[ Q_i = \int_{-\infty}^{+\infty} \mathcal{C}_\varepsilon \left[ \frac{k_i - \rho_i z}{\sqrt{1 - \rho_i^2}} \right] \phi(z)dz. \] (15)

Now, to determine the default barrier of the \( i^{th} \) firm we need to solve in \( k_i \) the equation (15).

We can easily extend the previous developments by introducing in expression (1), \( m \) independent factors \( Z_1, ..., Z_m \). Thus, we get:

\[ V_i = \sum_{k=1}^{m} \rho_{ik}Z_k + \omega_i \varepsilon_i, \] (16)

with \( \sum_{k=1}^{m} \rho_{ik}^2 \leq 1 \), we consider the same assumptions as above. In this context, equation (6) becomes:

\[ V_i = \rho_{i1}Z_1 + \rho_{i2}Z_2 + ... + \rho_{im}Z_m + \varepsilon_i \sqrt{1 - \rho_{i1}^2 - \rho_{i2}^2 - ... - \rho_{im}^2}, \] (17)

and the correlation between \( V_f \) and \( V_j \) can be expressed as:

\[ \text{Corr}[V_f, V_j] = \rho_{f1}\rho_{j1}E[Z_1^2] + \rho_{f2}\rho_{j2}E[Z_2^2] + ... + \rho_{fm}\rho_{jm}E[Z_m^2], \] (18)

which implies that:

\[ \text{Corr}[V_f, V_j] = \rho_{i1}\rho_{j1} + \rho_{i2}\rho_{j2} + ... + \rho_{im}\rho_{jm}. \] (19)

Thus, in an m-factor model, the conditional probability of \( V_i \leq k_i \) knowing the realizations of \( Z_1 = z_1, Z_2 = z_2, ..., Z_m = z_m \), is given by:

\[ P_i(z_1, z_2, ..., z_m) = \mathcal{C}_\varepsilon \left[ \frac{k_i - \rho_{i1}z_1 - \rho_{i2}z_2 - ... - \rho_{im}z_m}{\sqrt{1 - \rho_{i1}^2 - \rho_{i2}^2 - ... - \rho_{im}^2}} \right]. \] (20)
Here, the random variables $Z_1, Z_2, \ldots, Z_m$ are independent, thus the unconditional default probability $Q_i$ can be expressed as:

$$Q_i = \int_{R^n} P_i(z_1, z_2, \ldots, z_m) \prod_{k=1}^{m} \phi_k(z_k) dz_1 dz_2 \ldots dz_m,$$  \hspace{1cm} (21)

where $\phi_k$ denotes the probability distribution associated to each random variable $Z_k$. As in a one-factor model, we need to solve the equation (21) in $k_i$ in order to determine the default barrier for each firm $i$.

4 The Normal Inverse Gaussian (NIG) distribution

As mentioned in our introduction we are going to use NIG distributions in the factor models described previously. In this section we present the main properties of the NIG distribution.

The NIG distribution was introduced to investigate the properties of the returns from financial markets by Barndorff-Nielsen (1997)\textsuperscript{11}. This distribution has the remarkable property of being able to represent stochastic phenomena that have heavy tails and/or are strongly skewed. The NIG distribution is not confined to the positive half axis. The distribution is an obvious candidate for financial data for which Gaussian family often underestimates random variation, even after passing to logarithms of for example, prices of stocks and other securities. With the Normal Inverse Gaussian distributions we have at our disposal distribution which can be flexibly adapted to many different shapes. This distribution is characterized by 4 parameters $(\alpha, \beta, \mu, \delta)$. The parameter $\alpha$ is related to steepness, $\beta$ to symmetry, and $\mu$ and $\delta$ respectively to location and scale. The $NIG(\alpha, \beta, \mu, \delta)$ Probability

\textsuperscript{11}Since then, applications in finance have been reported in several papers, both for the conditional distribution of a GARCH model (Jensen and Lunde (2001); Forsberg and Bollerslev (2002); Venter and de Jongh (2002)) and for the unconditional distribution (Prause (1997); Rydberg (1997); Bolviken and Benth (2000); Lillestol (1998-2000-2001-2002); Venter and de Jongh (2002)).
Density Function is given by:

\[ NIG(x; \alpha, \beta, \mu, \delta) = a(\alpha, \beta, \mu, \delta)q\left(\frac{x - \mu}{\delta}\right)^{-1}K_1\left(\delta\alpha q\left(\frac{x - \mu}{\delta}\right)\right)e^{\beta x}, \]

(22)

with:

\[ q(x) = \sqrt{1 + x^2}, \]

(23)

and,

\[ a(\alpha, \beta, \mu, \delta) = \pi^{-1}\alpha \exp(\delta\sqrt{\alpha^2 - \beta^2 - \beta\mu}). \]

(24)

Here \( K_1 \) is a modified Bessel function of the third kind with index 1 defined as:

\[ K_1(x) = x \int_{1}^{\infty} \exp(-xt)\sqrt{t^2 - 1}dt. \]

(25)

The necessary conditions for a non-degenerated density are \( \delta > 0, \alpha > 0 \) and \( \frac{\lvert \beta \rvert}{\alpha} < 1 \).

The Moment Generating Function of a NIG-distributed random variable \( X \) is equal to:

\[ M_X(u) = \exp(u\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2})). \]

(26)

From equation (26) we can derive:

\[ E[X] = \mu + \delta \left(\frac{\beta}{\gamma}\right), \]

(27)

\[ Var[X] = \delta \left(\frac{\alpha^2}{\gamma^3}\right), \]

(28)

\[ Skew[X] = 3 \left(\frac{\beta}{\alpha}\right) \left(\frac{1}{(\delta\gamma)^{1/2}}\right), \]

(29)

\[ Kurt[X] = 3 \left(1 + 4 \left(\frac{\beta}{\alpha}\right)^2\right) \left(\frac{1}{(\delta\gamma)}\right), \]

(30)

with \( \gamma = \sqrt{\alpha^2 - \beta^2} \). From equations (29) and (30), and since \( \frac{\lvert \beta \rvert}{\alpha} < 1 \), we obtain some properties of the kurtosis and the skewness:

\[ Kurt[X] > 3, \]

(31)
and,

\[ 3 + \frac{5}{3} (\text{Skew}[X])^2 < \text{Kurt}[X]. \]  

Thus, there exists a bound, as illustrated in figure (4), on the skewness relatively to the kurtosis. We observe that the tails of a NIG distribution is always heavier than those of a Gaussian distribution. When \( \beta = 0 \) we obtain symmetric distributions. The Cauchy distribution is got for \( \alpha = 0 \) and the Normal distribution appears as a limit case for \( \alpha \to \infty \).

![Figure 4: Bound for the skewness relative to the kurtosis for NIG-distributions.](image)

An important property of the Normal Inverse Gaussian distribution is its behavior under convolutions. Indeed, if \( X_1 \) and \( X_2 \) are independent and distributed respectively as \( NIG(\alpha, \beta, \mu_1, \delta_1) \) and \( NIG(\alpha, \beta, \mu_2, \delta_2) \). Then \( X_1 + X_2 \) is a \( NIG(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2) \).
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Figure 5: PDF of zero-mean and unit-variance NIG-distributions.
In order to estimate the parameters from a given sample $x_1, \ldots, x_n$ of NIG-distributed random variables, we use the so-called method of moments. Then the mean, the variance, the skewness and the kurtosis are replaced by their empirical sample versions and the equations 27, 28, 29, and 30 are solved for $\alpha$, $\beta$, $\mu$ and $\delta$. The main advantage of this estimation methodology is that we can fix the mean and the variance of the distribution respectively to zero and one, as required in the factor model theory, and sensitize the kurtosis and the skewness. It is possible to build a lot of different types of NIG-distributions with zero-mean and unit-variance, see figure 5.

5 Calibration and pricing

In this section we describe an approach to valuing non-standard CDO tranches using a one-factor model and the implied distribution of the common factor derived from standard tranche market. As previously mentioned, the market-standard valuation methodology for synthetic CDO tranches employs the Gaussian one-factor model. The model takes as inputs:

- the underlying single-name CDS spreads in the reference portfolio,
- a recovery rate assumption for each reference name.

And the outputs of the model are:

- implied compound correlations.

We have illustrated in section 2 that the tranches’ implied compound correlations are not the same across the structure of a standard CDO. The purpose of our framework is to match market data by keeping the same correlation level for each tranche. In other words we want to correct “the correlation smile” by using the same portfolio loss distribution to price the five standard tranches of an index.

In the economic theory, the common factor $Z$ reflects the general state of the business cycle. Thus, instead of the implied compound correlations, we want to determine the general state of the business cycle which is implied by the CDO market. Our model takes as inputs:

- the underlying single-name CDS spreads in the reference portfolio,
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- a recovery rate assumption for each reference name,
- a pairwise correlation assumption between the underlying single-name CDS,
- as in the Gaussian one-factor model we use a Gaussian distribution for the idiosyncratic risk of each underlying single-name CDS.

And the output of our model is:

- the implied zero mean and unit-variance NIG-distribution of the common factor.

In our methodology the correlation between the underlying assets of the portfolio becomes an input. We will not discuss in this paper the pairwise correlation estimation because with our methodology, each user is free to choose his own method of correlation estimation and the assumptions concerning the recovery rates. We are now going to describe how we calibrate our model by using the price market quotes and then how we can price a non-standard CDO tranche.

In a first step, we study the sensitivity of each tranche’s spread (or upfront) relatively to the skewness and the kurtosis of the NIG-distributed common factor $Z$. As example, we use the iTraxx market quotes and composition of the 28th of April 2005 (see table 1 section 2). We assume that the average pairwise correlation of the underlying assets of this index is 20%. Then, using the properties of factor models and of NIG distributions previously discussed in this paper we make varying the skewness and the kurtosis of the NIG-distributed common factor and calculate the spread (or upfront) of each tranche for a 5, 7 and 10 years maturity. We illustrate our results in figure 6, 7 and 8. The dashed lines represent the market spreads (or upfront) of the tranches. We see that the sensitivity of the spread is not the same for each tranche. We observe that for a given maturity, each tranches’ spread (or upfront) has very different behavior. Theses differences of behavior between tranches in function of the distributions of the common factor $Z$ are the key-points of our framework. Now, we have to calibrate our NIG one-factor model.

\[\text{We mainly use the methodology proposed by Andersen, Sidinius and Basu (2003) and based on equity returns correlation and Frobenius norm to determine the factor loadings.}\]
Figure 6: Tranches’ spreads (maturity: 5 years) in function of the skewness and the kurtosis of the common factor $Z$. 
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Figure 7: Tranches’ spreads (maturity: 7 years) in function of the skewness and the kurtosis of the common factor $Z$. 
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Figure 8: Tranches’ spreads (maturity: 10 years) in function of the skewness and the kurtosis of the common factor $Z$. 
The second step of our algorithm extracts for each tranche the couples skewness/kurtosis that permit to match market quotes. Then, in a final step, we take the intersection of the previous solutions and obtain\textsuperscript{13} the best couple skewness/kurtosis that match market spread (or upfront) for each tranche. Thus, we determine the implied zero-mean and unit-variance NIG-distribution of the common factor.

We show, in table 2, the results of our calibration that match very well the market quotes for each maturity.

<table>
<thead>
<tr>
<th>Maturity: 5 years</th>
<th>Market</th>
<th>NIG one-factor model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tranche</td>
<td>Upfront(%)</td>
<td>Spread(bp)</td>
</tr>
<tr>
<td>0-3%</td>
<td>27.60</td>
<td>500</td>
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<tr>
<td>3-6%</td>
<td>0</td>
<td>171.50</td>
</tr>
<tr>
<td>6-9%</td>
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<td>53</td>
</tr>
<tr>
<td>9-12%</td>
<td>0</td>
<td>27</td>
</tr>
<tr>
<td>12-22%</td>
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<td>16</td>
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<tbody>
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<td>Tranche</td>
<td>Upfront(%)</td>
<td>Spread(bp)</td>
</tr>
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<td>0-3%</td>
<td>43.45</td>
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<tr>
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<td>52</td>
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<tr>
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<td>32.50</td>
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</table>

<table>
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<th>NIG one-factor model</th>
</tr>
</thead>
<tbody>
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<td>Tranche</td>
<td>Upfront(%)</td>
<td>Spread(bp)</td>
</tr>
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<tr>
<td>12-22%</td>
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<td>53.50</td>
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</table>

Table 2: Upfronts and spreads calculated with an average pairwise correlation of 20\% and a NIG one-factor model versus market quotes.

We illustrate in figure 9 the difference between the implied common factor and a Gaussian common factor and also the portfolio loss distribution for each maturity.

\textsuperscript{13}We minimize the differences between our results and the market quotes.
CDOs pricing and factor models: a new methodology using Normal Inverse Gaussian distributions.

Figure 9: Implied common factor (blue) versus Gaussian common factor (red) and portfolio loss distribution.
Thus, to value a Euro non-standard CDO tranche with maturity $T$ we propose the following procedure:

1. determine the pairwise correlation level of the index (maturity $T$) underlying single-name CDS,

2. use our algorithm to determine the implied NIG-distributed common factor given this correlation level,

3. determine the correlation level (using the same method than in step one) of the studied CDO’s underlying single-name CDS portfolio,

4. run the corresponding NIG one-factor model to estimate the spread of the studied non-standard CDO tranche.

6 Conclusion

In this paper we have introduced a new methodology for the valuation of CDOs. By using NIG-distributions in the factor models framework we have proposed an alternative way to compute the spread of a tranche. In the market standard methodology, the distributions used in the factor model are fixed and the implied correlation of a tranche is calculated under these assumptions. We have used an opposite approach: we fix the correlation of the portfolio and the distribution of the idiosyncratic risk of each firm and then we determine the implied distribution of the common factor in the model.

This methodology appears to be theoretically more accurate. All tranches are priced with the same correlation assumptions and the distribution of the common factor reflects the “market’s vision” of the general state of the economy. We have illustrated that our methodology provides strong results. Provided that we use the same method of correlation estimation for all underlying portfolios of CDOs and for the calibration of the model, we can use the implied distribution of the common factor to price all kinds of non-standard tranches of CDOs.

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References


