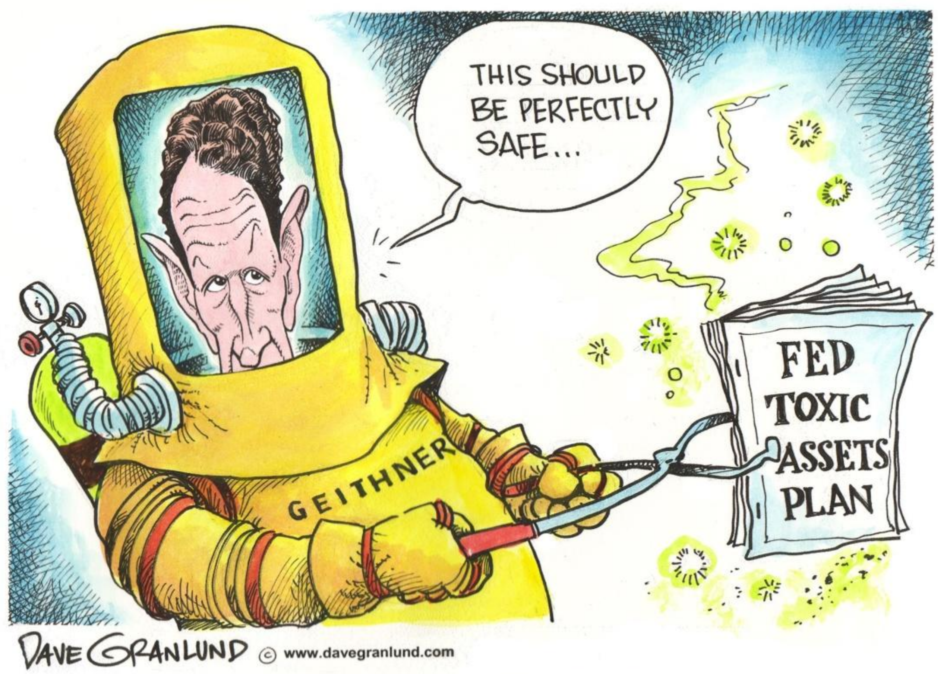
Analysis of the Public/Private investment Plan (PPIP) in the US and Formulation of Alternative Structures for Comparative Analysis



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Summary

This article focuses on one aspect of policy responses to the 2008 crisis in the US. It analyses the development and structure of the "Public/Private Investment Plan" PPIP and "Public/Private Investment Fund" PPIF as a means of relieving financial institutions of their troubled assets to restore liquidity in financial markets and compares with an alternative structure whereby the government would become a market maker on the troubled assets possibly traded at a refined level of granularity.

The interest of the article is not limited to the topic covered, but also extends to the methodological as illustrated in the use of the concept of BICs[i] to address a non-trivial analytical and computational problem.

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Introduction

As markets froze in the aftermath of Lehman Brothers' collapse and the government takeover of AIG in September 2008, interbank dealing fell to anemic levels. Banks suddenly felt they had no understanding of each other's balance sheet in order to appreciate credit worthiness. The US Treasury in conjunction with the Federal Reserve took a number of emergency steps in order to restore confidence in the markets and restore banks ability to lend to real economy businesses.

One such measures floated since September 2008 in various versions and finally announced on March 23, 2009 was the Treasury Public Private Investment Plan. When announced, the Geithner[ii] "Public-Private Investment Plan"[iii] or "Public-Private Investment Funds" (PPIP/PPIF) outlined the establishment of 50/50 equity owned public-private funds coupled with low interest government loans for 85% of each fund's balance sheet. These funds would then be mandated to buy troubled assets.

The plan was an intermediary stage in an evolving search process that began in September 2008. **One of the thorniest issues was how to fairly price the troubled assets**. On October 1, 2008, Warren Buffett[iv] was suggesting a solution might be that the government requires banks to first go sell part of the assets to investors in the market. The government would then come in and match the prices paid by outside investors. The effort was continued in the new Obama administration and the term "Bad bank" became the name commonly used to refer to the government entity buying the assets.

Since then, it has undergone several adjustments on the size of equity and debt provided by the government and the size of entities that would qualify for managing PPIF. At the end of April 2009[v], the proportion of debt in PPIF seemed to have been lowered to 50% and adjustments were made to the minimum size of the funds. However, it seems that the basic structure of the plan (50/50 Government/Private Investors Equity + Low Interest Government Funded Debt) will remain as is when it is finally launched in the summer of 2009.

In the first part of this paper we will explain how, because of its structure, the PPIP is bound to embed subsidies so as to give rational investors' participating in the plan an incentive to bid the assets higher than their expected value.

We will show that the subsidies come at least both from the low interest loans provided to the funds bidding on the assets as well as from the non-recourse nature of those loans.

In the second part, we show how an alternative structure, in the form of the government setting itself up as a market maker on the troubled assets bought at a refined level of granularity would have effectively restored market liquidity on the assets, while creating an inventory which tended to be bought below its expected value. In this scheme depending on the scope of the problem, the assets may temporarily be held and aggregated in a trust.

One of the most prominently advocated alternatives, in the form of nationalization of troubled financial institutions is intentionally not discussed here.

Subsidies in the PPIF

First Underlying Subsidy: The interest rate discount on debt

The first underlying subsidy is straightforward and has to do with the discount on interest rate debt. A simple example may illustrate it best:

"Suppose you want to buy a toxic asset T. You think that when you dispose of it in a year it will be worth $110. How much will you be willing to pay for it today? It all depends on the cost of borrowing. If you can get a 0% loan you can pay up to $110. If the interest rate is 10%, you will not pay more than $100 for T. So the lender who lends you money at 0% when the market on such loans is 10% is giving you a subsidy that you may share with the asset T seller. Furthermore lending such subsidized money to more than one potential buyer will ensure through competitive forces that the bulk of the subsidy is passed on to the seller.

Another way of seeing the subsidy is by arguing that interest payments on corporate debt are a business cost that reduces the return on equity. Therefore reduced interest on debt is a form of subsidy to equity investors that can be passed on in the price of purchased assets.

Second Underlying Subsidy: "The Geithner Put"

The idea of giving out a put in the setup of PPIFs comes from the fact that equity investments in limited liability entities is always equivalent to having a long put option since losses are limited to the amount invested.

What has been called the "Geithner Put" is a comparatively sophisticated analytic argument. It was first posted in Blogs such as Nemo Publius[vi] and Paul Krugman[vii] soon after the plan was announced. It is based on the non-recourse nature of the loan in PPIFs. The example provided by Krugman to explain this idea as follows:

"Suppose that there's an asset with an uncertain value: there's an equal chance that it will be worth either 150 or 50. So the expected value is 100. But suppose that I can buy this asset with a non-recourse loan equal to 85 percent of the purchase price. How much would I be willing to pay for the asset? The answer is, slightly over 130. Why? All I have to put up is 15 percent of the price — 19.5, if the asset costs 130. That's the most I can lose. On the other hand, if the asset turns out to be worth 150, I gain 20. So it's a good deal for me. Notice that the government equity stake doesn't matter — the calculation is the same whether private investors put up all or only part of the equity. It's the loan that provides the subsidy. And in this example it's a large subsidy — 30 percent."

We will provide here a more general analytical framework for understanding this example in a way that can enable more general conclusions for reuse in other settings.

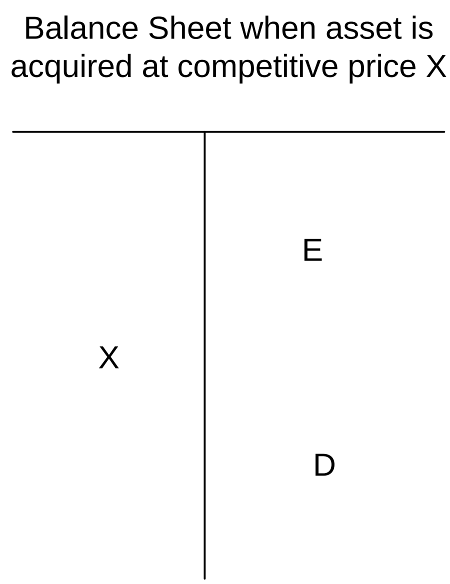
Analytical Framework

We seek to describe how funding through debt shapes the price of assets to be acquired in competitive markets. Suppose we have an asset S whose value at disposal time is unknown and we'd like to acquire it at price X.

The General Case

We decide to finance the acquisition through a combination of equity and non-recourse debt. The balance sheet of the structure created for that purpose is:

Figure 1: Sample balance sheet



In a competitive environment,

- What would be the equilibrium price of X?
- How does it compare with the expected value of the asset E(S) ?

The unleveraged buyer would equally enjoy profits and suffer losses. Under traditional expectations based pricing, if we assume the interest rate discount insignificant over the horizon at which the asset is divested, the price would equal its expected value. As such:

$$X = E(S), i.e: E(S - X) = 0.$$

However, for the leveraged buyer who is financed by a non recourse loan, their loss is floored by their equity investment. As such, if we note e_r the ratio of equity over equity plus debt; indeed e_r varies between 0 and 1. A value of 0 means the asset is acquired with no equity and a value of 1 means the acquisition is entirely financed through equity.

A price too high would lead to an expectation of loss while a price too low would lead to an expectation of free lunch. Competitive pressures push prices up since in a competitive market the asset would go to the highest bidder and each rational bidder can only go as high as an expectation of profit allows.

The payout at divestment is thus $Max\left(-e_r X,\left(S-X\right)\right)$ and the maximum equilibrium price the highest bidder with the given financial structure would possibly be expected to bid is implied by the equation

$$E\left(Max\left(-e_r X,\left(S-X\right)\right)\right)=0 \qquad (1)$$

Notice that

$$Max\left(-e_r X,\left(S-X\right)\right)=\left(S-X\right)+Max\left(\left(1-e_r\right)X-S,0\right) \quad (2)$$

leads to

$E\left(\left(S-X\right)\right)\leq E\left(Max\left(-e_r X,\left(S-X\right)\right)\right)=0$, hence

$$E\left(S\right)\leq X \qquad (3)$$

And

$$X=E\left(S\right)+E\left(Max\left(\left(1-e_r\right)X-S,0\right)\right) \quad (4)$$

That means the excess price over the expected value paid is equal to the price of the put option on S struck at $\left(1-e_r\right)X$. This is indeed what was termed as “The Geithner Put”. The problem now is how to value this put. Depending on the assumptions made on the distribution of S, this put’s value can vary from zero to any arbitrarily large number. We consider below a further simplified tri-states case scenario that helps better understand the Krugman example.

The Tri-States Case

Let’s suppose the values of S at divestment time are constrained in a range such that $S_l \leq S \leq S_h$, indeed X solving (1) would be similarly constrained $S_l \leq X \leq S_h$. As such (2) would narrow the boundaries to

$$E\left(S\right)\leq X \leq S_h \qquad (5)$$

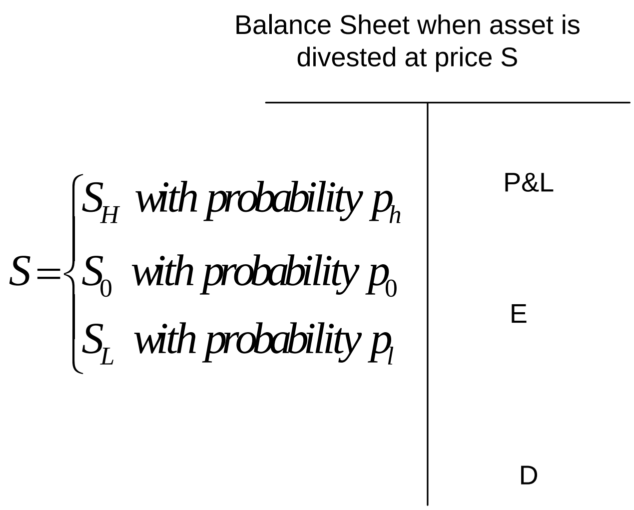
Note that the proportion of individual investors’ holdings in the total equity pool is irrelevant if profit and losses are distributed in the same proportion.

Our analytical derivations will be focused on solving equation (1).

This requires specifying the range of values S may take and their associated probabilities. The range of values may be large or small depending on the type of analysis one wishes to perform and the constraints one must calibrate to.

For the examples we wish to investigate, it is enough to consider that the range of values of S is limited to 3 values (S_l, S_0, S_h) and the associated probabilities (p_l, p_0, p_h) are calibrated by the expected value taken as E and the standard deviation given as σ.

Figure 2: Sample 3-States Balance Sheet



Therefore solving the system of equations and inequalities $\begin{cases} 0 \le p_l, p_0, p_h \\ p_l + p_0 + p_h = 1 \\ p_l S_L + p_0 S_0 + p_h S_H = E \\ p_l S_L^2 + p_0 S_0^2 + p_h S_H^2 = \sigma^2 + E^2 \end{cases}$

yields $P = \begin{pmatrix} p_l \\ p_0 \\ p_h \end{pmatrix} = \begin{pmatrix} \dfrac{(S_0 - E)(S_0 + S_h) + \sigma^2}{(S_0 - S_l)(S_h - S_l)} \\ 1 - \dfrac{(S_0 - E)(S_l + S_h) + \sigma^2}{(S_0 - S_l)(S_h - S_0)} \\ \dfrac{(S_0 - E)(S_0 + S_l) + \sigma^2}{(S_h - S_l)(S_h - S_0)} \end{pmatrix}$ with additional inequality constraints[viii]

If we note $M = \begin{pmatrix} Max(-e_r X, (S_l - X)) \\ Max(-e_r X, (S_0 - X)) \\ Max(-e_r X, (S_h - X)) \end{pmatrix}$ we have $E(Max(-e_r X, (S - X))) = M^t P$

We assume $S_l \geq 0$. Since we have $Max(-e_r X, (S - X)) = -e_r X \text{ if } X > \frac{S}{1-e_r} \text{ Or else } (S - X)$, we can compute the table of solutions to the equation $M^t P = 0$ in the table below.

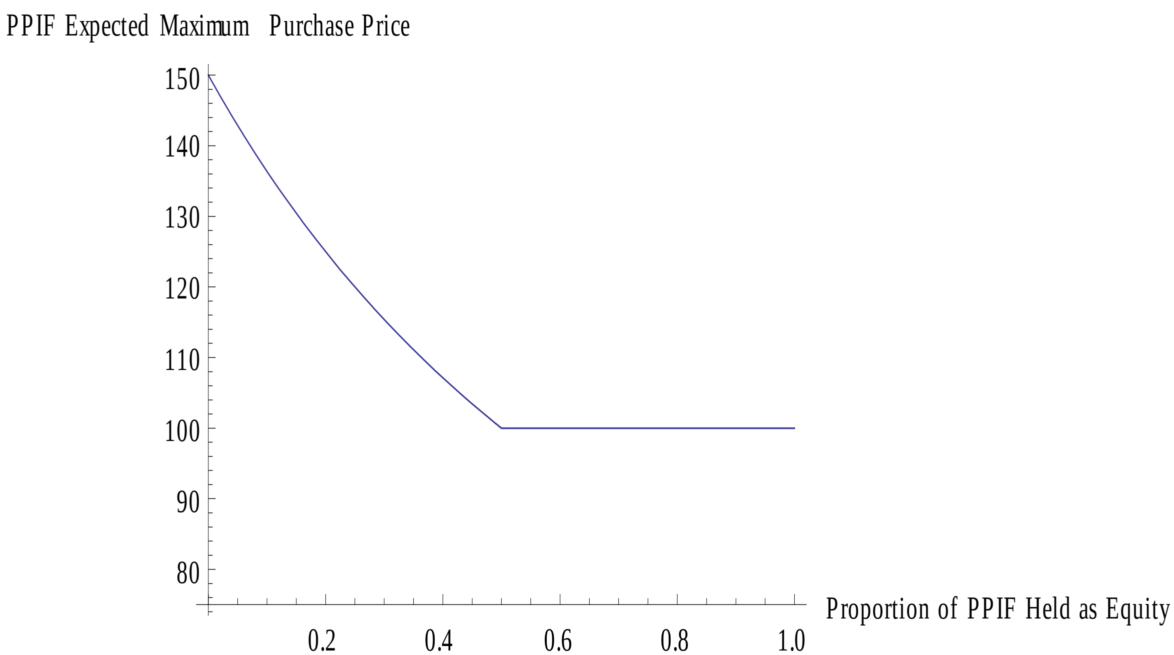
Table 1: Range of Solutions for X

Assumed Range of X	$Max(-e_r X, (S_l - X))$	$Max(-e_r X, (S_0 - X))$	$Max(-e_r X, (S_h - X))$	X Solving the equation
$S_0 \leq X \leq \frac{S_l}{1-e_r}$	$(S_l - X)$	$(S_0 - X)$	$(S_h - X)$	$X = S_0$
$S_0 \leq \frac{S_l}{1-e_r} < X < \frac{S_0}{1-e_r}$	$-e_r X$	$(S_0 - X)$	$(S_h - X)$	$p_l = \frac{(S_0 - E)(S_0 + S_h) + \sigma^2}{(S_0 - S_l)(S_h - S_l)}$ $X = \frac{(S_0 - p_l S_l)}{(1-(1-e_r) p_l)}$
$\frac{S_0}{1-e_r} \leq X \leq S_h$	$-e_r X$	$-e_r X$	$(S_h - X)$	$p_h = \frac{(S_0 - E)(S_0 + S_l) + \sigma^2}{(S_h - S_l)(S_h - S_0)}$ $X = \frac{p_h}{(p_h + e_r(1-p_h))} S_h \leq S_h$

We use this derivation to compute the results of the Krugman example

The Krugman example seems to have been particularly influential with the administration in pushing it towards less generous terms for investors. As of 04/30/09, Risknews[ix] reported that the subsidized government loan had been lowered from the initial 85% to now 50% of the balance sheet of the funds. Plugging these numbers in the Krugman example changes the originally computed 130.43 to 100. In this case the “Geithner Put” has zero value.

Figure 3 : Expected Break-Even price by percentage of PPIF held as equity when such equity is equally distributed between government and private investors ; 15% leads to 130.43, 50% or higher lead to 100.



An Investor's Perspective

From an investor's perspective, the analysis above illustrates two dimensions of a fallacy which we argue is at the core of risk management failures at the root of the 2008 crisis. Exposing theses fallacies could even be used to prove the argument opposite to the one being conveyed perhaps rightly, with respect to the extent of the subsidy provided to financial institutions.

The first typical problem exemplified here is that at its core, by basing its reasoning on expectations assessments, it misses one the most relevant elements of investors preferences based on resources constraints; many investors have already lost a lot, and unlike the government have limited liquidity and time frames and it would not be unreasonable for them to demand more to cushion the risks to what they've got left. Let us stretch a bit to drive the point home:

Suppose Mr. Krugman's net worth is $100 million and we flip a coin with equal probability of head and tail. If it falls on head, Mr. Krugman wins and receives $1 billion; Otherwise, he loses everything he has got and pays out his net worth of $100 million. Will he take the bait?

I do not know the structure of Mr. Krugman's risk preferences but many investors might rationally conclude that they cannot afford such a risk, even though their expected gain here would be $450 million. This is not a mere thought experiment; for an actual adverse outcome, Google "Victor Niederhoffer"; check his Wikipedia entry on the "1997 losses" section. A logical continuum of duress pricing would rationally lead investors to request excess incentive beyond fair value expectations to reflect the relative constraints on respective resources. When the deal goes south, the investor credit might get impaired, raising their cost of capital for subsequent activities. The cost of provisioning for such contingencies may account for requesting positive expected excess returns. Indeed that's why not every investor will be rushing to start a PPIP. A useful metaphor of this issue is provided by Nassim Taleb, author of "The Black Swan" when he says:"Never *cross a river* because it is on *average* 4 *feet deep*."

In the aftermath of the failure of LTCM, when most partners in the fund lost almost all of their net worth, Warren Buffett had a more colorful example to make this point to MBA students. Instead of losing your net worth, he would dramatize by saying that when you lost, the single bullet of a pistol would kill you and would ask, excluding himself, for how much any listener would want to play the game, and everyone in the audience would just shrug.

The analytical bias described above is one instance of what can be called the *fallacy of expectations based risk management*. Many of the firms that go under in every financial crisis make the same mistake, often advised by the smartest people. *They make their investment decisions based on expectations profiles without full appraisal of the sustainability of downside scenarios* that even higher order metrics such as standard Value At Risk fail to capture. Macro-economists are most prone to this type of oversight as they tend to view the world from a sovereign's perspective of unlimited access to liquidity and unlimited life span.

More generally, this fallacy is also at the heart of derivatives pricing and hedging methodologies. Indeed, a derivatives contract can be viewed as an uncertain future payoff that is a function of as yet unobserved values of underlyings on a future timeline. The underlying future values may be spanned along multiple scenarios. Standard pricing algorithms then compute the derivatives contract price as a discounted average of the contract's payoff for each scenario weighted by the assumed likelihood of the scenario. The best dynamic hedging strategies then work by dynamically taking opposite positions in underlyings or their substitutes in proportions that offset changes in the estimated values of the derivatives contract over time.

The second typical problem exemplified in this reasoning is that by conveniently ignoring questions over the uncertainty[x] in the probabilities used, the analytic framework contributes to perpetuate a risk management philosophical failure that may trigger many subsequent financial bankruptcies long after this crisis and recession are weathered.

Recognizing the dangers lying in such analytic frameworks, this author strenuously argues w that robust risk management strategies ought to be based on directly trading contracts that encapsulate and lock-in future scenarios expectations. How does this work? Let's further consider the "Geithner Put" example above.

Suppose now that rather than working with expectations as done in Krugman's example, we could

- buy a contract that will payout *50%Min (15%asset purchase price , (asset purchase price-50)*) if the price of the asset that would be lost in the adverse scenario when the asset ends up being worth only 50 and
- finance it by selling a contract that will payout *50% (150 - asset purchase price)* if the asset ends up being worth 150.

If we use the "50/50" likelihood scenarios as described to price these contracts, then if the toxic asset purchase price is 130 or less, the contract bought would be cheaper than the contract sold, resulting in a net and present risk free profit without any future exposure.

The hedging contracts encapsulating risk described in the example are what are usually called Arrow Debreu securities (ADS). In this framework, we see that the un-hedged investor is actually selling the first ADS and buying the second one as well while deciding the price at which to immediately buy the related troubled assets.

The notions of probability and expectations based on often questionable statistical inference as used in time and resource constrained risk management are dangerous because they have no meaning in themselves if there is no party in the market willing to trade ADS on the assumed probabilities. ADS here not only do encapsulate the rational expectations of investors and the constraints on the resources of those who trade them, they stress more emphatically the need to look for replicating alternatives to offset the risks taken on. When ADS or substitute do exist, their quoted prices would then serve as basis for inferring actionable pricing probabilities with hedging value.

ADS are good pedagogical tools when we deal with relatively simple scenarios such as the ones in the example above. In real life, outcomes play out in multiple periods time frames that make these ADS computationally impractical. Recognizing these limitations, the author has introduced the concept of Basis Instrument Contract (BICs)[xi] as a more viable alternative for real world applications on a larger scope.

While it may not always be possible to have available hedging Basis Instruments Contracts that ensure total elimination of risk, responsible risk management practice must mandate that risky contracts be hedged so that in the residual exposure, the worst possible scenarios be reduced to a scope that does not severely strain available institutional resources. The spirit of such a practice is in part provided in FAS 133 &157 as well as IAS 39 with the minimum required percentage of hedged and un-hedged parts of derivatives contracts. However these accounting rules as currently written are shrouded in complexities and imprecision that can render them ineffective using merely currently widespread analytical tools. The more rigorous mathematical framework of BICs would seamlessly enable a more effective and efficient implementation of this spirit.

In the next section, we advocate out market making approach to restores liquidity on troubled or illiquid assets

An Alternative Market Making Approach

Intuition

Once the government decides to buy these distressed assets to provide liquidity, it ipso facto becomes a market maker and must quote prices like any rational market maker would. This price obviously depends on the price at which the asset last traded. This can be used as an anchor starting price from which to start bargaining. So requesting that banks first go out and sell at least a portion of those assets in the markets might be a good way to establish that anchor price as Mr. Buffet originally hinted. But this is not absolutely necessary; any other last traded price in the markets would be acceptable as well. In Mathematics, this is called seeding the algorithm. There are two key factors that would well help effectively reach the fair price :

Factor 1. The inventory of those assets held by the market maker. The general rule is the higher the inventory held, the lower the market maker bid prices. Conversely, the lower the inventory held, the higher the market maker asks. This is the driver of market asset prices fluctuations in response to supply and demand forces that a true fair value pricing method or algorithm should reflect. Their are many ways to do this[xii]. A simple model reflecting this reality would be to apply a **discount factor** on each next unit of asset bought from the price of the previous one. Conversely an **appreciation factor** may be applied on each next unit of asset bought from the price of the previous one.

Factor 2. This oft *under appreciated* factor in speeding up convergence towards fair market value of the security is the level of granularity or **refinement** of the unit security. The more granular the unit security, the faster the convergence towards fair price will be. That’s why stock markets, through unit stock granularity, work so well in revealing companies fair value.

If the discount factor reflecting factor 1 is too steep, it reflects a highly illiquid market. By contrast a high level of granularity reflective of Factor 2 helps the total price converge to where the market value actually should be while maintaining relatively narrow spreads on unit price quotes.

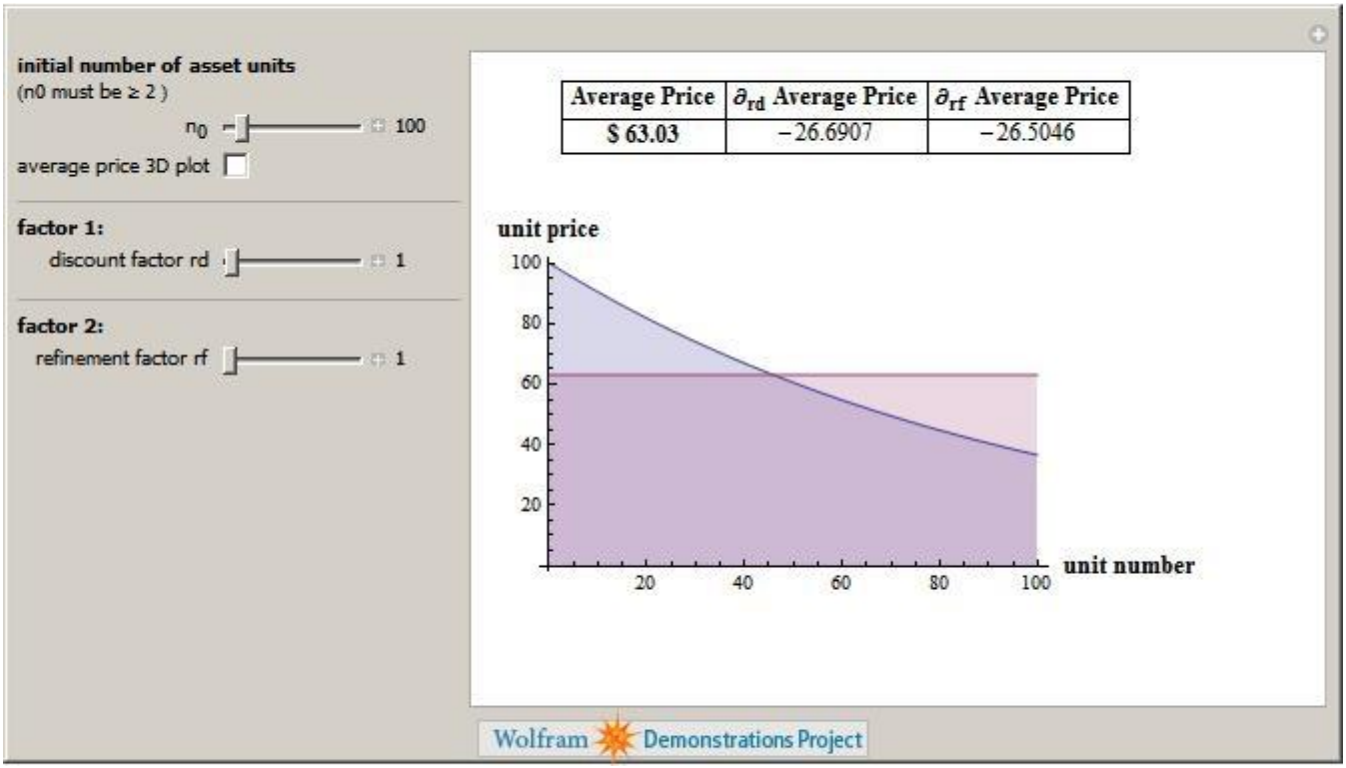
Furthermore, in the simplified version of this model, the sensitivity on factor 1 (rd) is structurally of a different order compared to the sensitivity on factor 2 (rf) : In this simplified model, the sensitivity on rd is polynomial while the sensitivity on rf is exponential.

An Example

Let’s look closely at how this matters in a simple example. Suppose Bank B wants to sell 100 units of toxic assets TB. It sells the first unit in the market at $1 and then liquidity dries up. If we adopt a price matching algorithm, the fair value of the remaining 99 units would be $99.

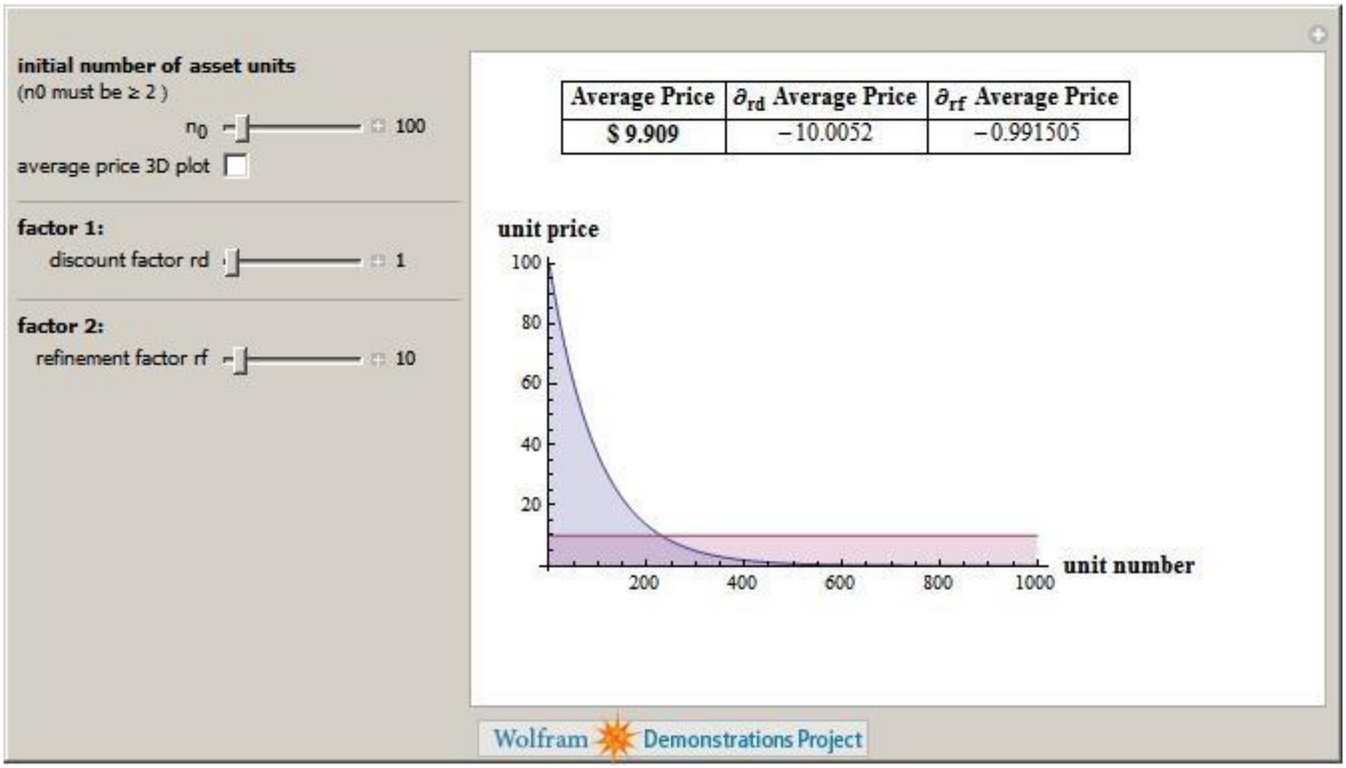
If we have a market maker whose price incorporates factor 1, a very generous pricing method for the seller could be for example requesting that every next unit bought would be at a 1% discount on the price of the preceding unit. By iterating the process on the 99 units, the fair-value of the remaining 99 units ends up falling to $63.03[3]

Figure 4



The starting unit price and the depreciation rate are of secondary importance as long as they are positive and non zero. In our example above, if every toxic asset TB is further subdivided into 10 units of toxic assets, the fair value of the remaining 99.9 units would fall even down to $9.91[4].

Figure 5



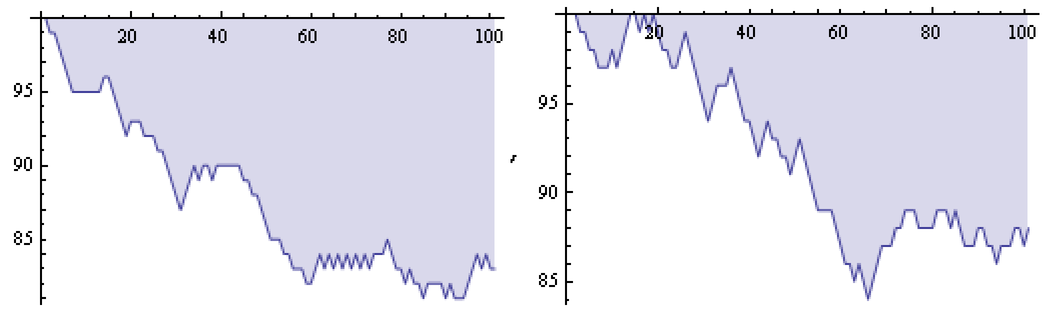
Does that mean that the value of the remaining assets is $63.03? Or is it then $9.91? Not necessarily and probably none of those. As the unit price falls, bargain hunters will step in and bid higher. This will reset the Government bid price higher or allow it to progressively unload its own inventory at a potential profit, using a similar ask pricing formula. If no bargain hunter feels confident enough to step in, even greater unit asset granularity will ensure that the average unit price paid by the government is close to zero, reflecting true market sentiment

As more bargain hunters step in, the markets will become liquid faster; the proportion of total assets held by the government Bad Bank will be smaller; government intervention in the markets will be minimized as it should be and Adam Smith's invisible hand will work its magic yet again. The example above shows that setting up, managing and winding down such a fair value system can be fairly straightforward and at minimal cost. This process will be further detailed below.

Detailed Analysis

In the previous section, we explained how a suitable mechanism or architecture for revealing PPIP assets prices is through a market making process of those assets split to a refined level of granularity. The model presented therein to illustrate our argument merely looked at the scenario where no private bargain hunters step in to compete on the purchase price of the assets. As we stressed in the article there are many possible and more likely scenarios. In fact the probability of the scenario considered may decrease exponentially fast to zero as the number of assets to be disposed of increases. Sample scenarios look more like those on figure 6.

Figure 6



In this section we deepen the analysis to all the alternative scenarios and help answer fundamental questions a decision maker may need to consider:

1. What inventory can we expect to hold at the end of a market making process?
2. How much can one expect this inventory to cost?
3. What would be the expected average unit inventory cost?
4. What is the likelihood of ending with no inventory?
5. What expected profit or loss would such no inventory scenarios entail?

In the process of making such estimations, the **BICs** framework will naturally emerge as the most efficient process for the most accurate estimates.

Analytic Framework

Analysis of the whole range of possibilities requires first describing the dynamics of the likelihood of alternative possibilities.

Incremental Possible Scenarios Representation

For each incremental unit asset disposition, the market maker may buy the unit asset if they are the highest bidder, sell the unit asset if it makes the lowest offer or see two third parties buy and sell if the settlement price falls in between. This incremental unit asset size may be chosen as the typical trade size that would cause price fluctuations. As such, for each incremental unit transacted, one can identify three possible scenarios:

- The high state case corresponding to the case where the market maker is making the lowest offer. In this case, the market maker sells the unit asset if there is any in their inventory.

- The middle state case corresponding to the case where the market maker is neither making the highest bid nor making the lowest offer. In this case, two third parties exchange the unit asset among themselves. *at the same price as the preceding transaction price*
- The low state case corresponding to the case where the market maker is making the highest bid. In this case, the market maker buys the unit asset.

One of the apparently simplifying assumptions of the previous section was that of a constant depreciation rate on each next unit bought. This approach is very effective and reasonable to illustrate how granularity leads to a more effective price discovery process for which discount rate is not a substitute. In practice however, the standard is to depreciate or appreciate on constant basis points units rather than as a percentage of the preceding unit asset price. Compared to the constant depreciation rate approach, this practice accelerates depreciations as any fixed non zero unit basis point eventually becomes larger than any fixed proportion of a shrinking quantity; likewise, the practice decelerates appreciations as any fixed non zero unit basis point eventually becomes smaller than any fixed proportion of an exploding quantity. Since we focused in the previous section on how a depreciating asset works, a constant depreciation rate analysis is in fact more refined.

Once we decide to study both appreciations and depreciations in the unit asset prices, it is simply more realist to work from a unit basis points shifts approach. *The incremental basis point move will be referred herein as d*. The extent to which a market is liquid may be represented by the width of the bid/offer spread, which for the government market maker studied here is *2d*. The larger the spread, the less liquid the market. An entity, governmental or otherwise that steps in to restore liquidity presumably does offers the minimum bid/offer spread on the unit underlying asset among all the other market participants.

These three incremental state possibilities may thus be effectively represented as:

- a basis point incremental increase compared to the price of the preceding unit asset transacted;

- a price that remains the same as that of the price of the preceding unit asset transacted or

- a basis point incremental decline compared to the price of the preceding unit asset transacted.

We can extend the number of possible states to more than three but that would not fundamentally change the analysis; a wider range of possibilities can simply be translated in this framework as further splitting the unit asset and considering the possible scenarios over correspondingly larger unit increments.

States probabilities distributions

As described above, at each stage, the transaction price of the incremental unit asset has three state possibilities corresponding to three probabilities of price evolution. Each one of these probabilities depends a priori on the path of the last traded incremental unit assets**.** A reasonable but expandable description of this dependence is to encapsulate it in the last traded incremental unit asset. When such a choice is made, the mathematical term used is to state that the incremental unit asset price is a Markov chain. Therefore, if p and q are two real functions on positive numbers whose values are constrained between 0 and 1, the probabilities of the corresponding 3 states may be described as:

- $(1-p(S_i))(1-q(S_i))$ for a basis point incremental increase compared to the price of the preceding unit asset transacted;
- $(1-p(S_i))q(S_i)$ for a price that remains the same as that of the price of the preceding unit asset transacted;
- $p(S_i)$ for a basis point incremental decline compared to the price of the preceding unit asset transacted,

where $0<=p(S_i),q(S_i)<=1$.

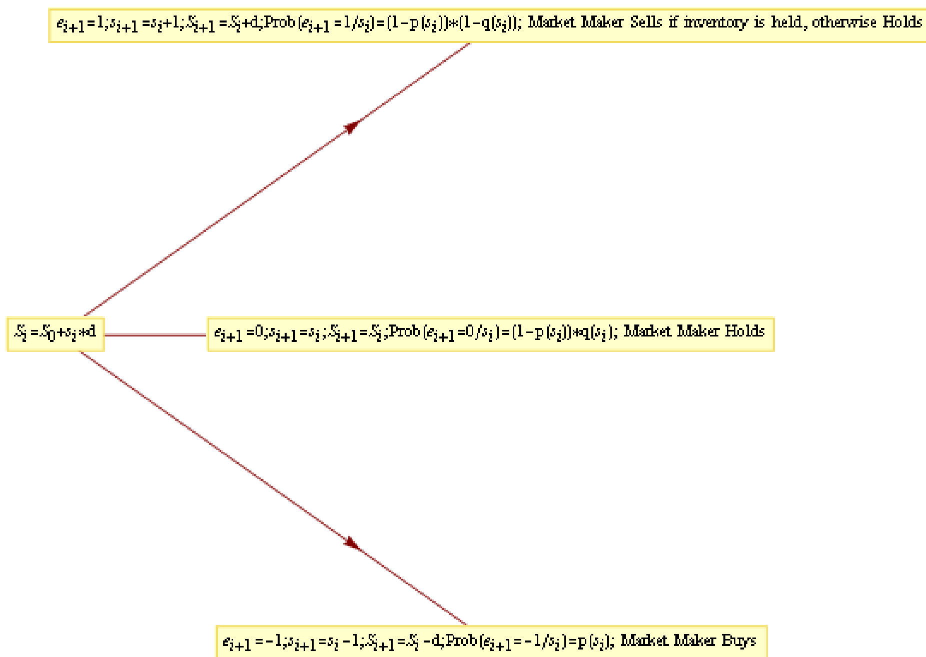
Realistic choices for the functions p and q must satisfy two basic constraints:

- For p, as Si goes further down the probability of continuing to go still down must decrease.
- For q, as Si goes further up the probability of continuing to go still up must decrease.

Note that choosing q = 0 reduces the future states possibilities from 3 to 2, an up and a down case.

Figure 7 below summarizes our analytical framework

Figure 7

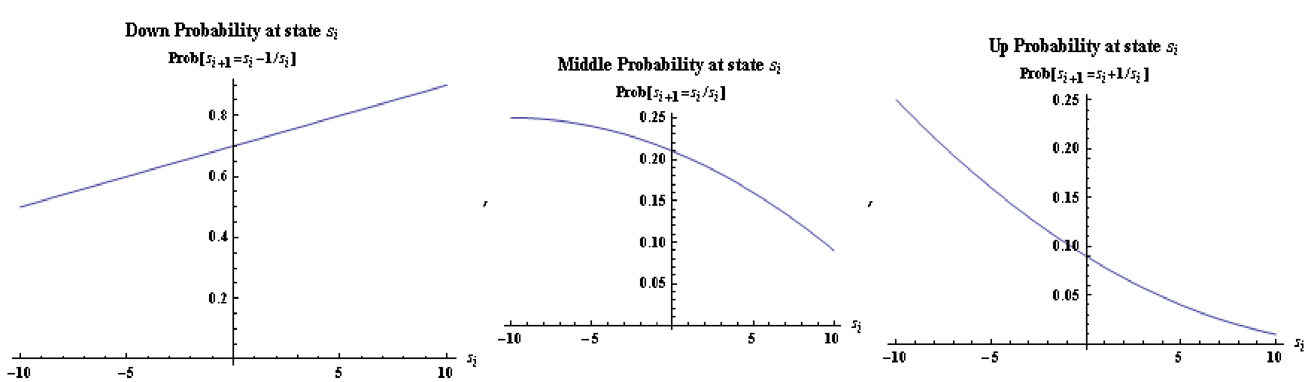


Incremental Asset Prices Scenarios

A canonical example of functions p and q satisfying the specified constraints

A simple way of defining p and q is to specify both of them as ***quadratic functions*** determined by the value for p and q at three different points. These values may be specified as the values associated with the minimum price of the unit asset, the maximum price of the unit asset and median unit asset price. The graphs below show sample down, middle and up state probabilities corresponding to p, q(1-p) and (1-q)(1-p) respectively.

Figure 8



The graphic representation makes intuitive sense as the down probability increases as the asset price increases and the up probability decreases as the asset price increases.

Underlying Selection

The underlying is the price S of the unit asset. The i-th unit asset to be transacted is denoted S_i .
The underlying can also be reduced to the variable s taking the value 1 if the unit price goes up, zero if t stays the same and -1 if it goes down. For the i-th unit asset to be transacted, the corresponding variable s is denoted s_i If we assume the under .The correspondence between S and s is given by $S_i = S_0 + s_i * d$; $s_i = (S_i - S_0)/d$;
$S_{i+1} = S_i + e_i$; e_i in {-1,0,1};

We assume the number of illiquid assets to be disposed of is n.

Reformulation in Derivatives Pricing Terms

Answering the questions of a decision maker translated as pricing derivatives contracts

Answering the five questions posed can be translated into the pricing of five different path dependent derivatives contracts whose payout depends on the incremental unit asset prices along a pseudo time scale indexed by the exchange of an incremental unit asset between a buyer and a seller.

Each one of the questions is answered when the the realized value of the underlying up to the n-th asset is known. If we view transacting the i-th asset as the i-th time increment and the sought answer as a payout that is a function of the realized values of the underlyings, then estimating the expected outcome before any of the assets has been transacted is equivalent to *pricing five different derivatives contracts* where there are no interest rates and hence the discount factor is taken to be one

The argument for working with expectations

The choice of expectations to generate estimates is justified in this instance by the fact that the government a priori has access to unlimited liquidity and will be there forever so that the results of the law of large number, i.e. expectations should be its guiding tool in making decisions. Indeed when we deal with private entities constrained both in time and liquidity, decisions should preferably be tied to instruments that price the likelihood of future events into present contracts. This fact is often lost on macro-economists when analyzing private investors choice, with sometimes devastating results. Indeed this is where BICs are most helpful as risk management tool and instruments.

Payout Functions & Mathematical Statement of Computational Objectives

Having reformulated the problem in derivatives pricing terms, the payout payment functions for each of the 5 questions are as follows:

1. The payout payment function whose expectation provides an estimate of the inventory we can expect to hold at the end of a market making process is $I_n (s_0,\ldots,s_n)$ defined recursively by:

$$I_{i+1}(s_0,\ldots,s_{i+1}) = (I_i(s_0,\ldots,s_i)-e_{i+1})^+;$$

We need to to compute $E_0(I_n (s_0,\ldots,s_n))$, the expectation at time i=0 of $I_n (s_0,\ldots,s_n)$

2. The payout payment function whose expectation provides an estimate of the amount spent to ensure liquidity on the n assets is $C_n (s_0,\ldots,s_n)$ defined recursively by:

$$C_{i+1}(s_0,\ldots,s_{i+1}) = C_i(s_0,\ldots,s_i)+$$

$$(S_0+s_{i+1}'*d)*((-e_{i+1})1_{\{}I_i(s_0,\ldots,s_i)_{>0\}}+(-e_{i+1})^+1_{\{}I_i(s_0,\ldots,s_i)_{<=0\}});$$

We need to to compute $E_0(C_n (s_0,\ldots,s_n))$, the expectation at time i=0 of $C_n (s_0,\ldots,s_n)$

3. The payout payment function whose expectation provides an estimate of the average unit inventory cost we can expect $M_n (s_0,\ldots,s_n)$ defined recursively by:

If $I_n = 0$, $M_n (s_0,\ldots,s_n) = 0$, else $M_n (s_0,\ldots,s_n) = C_n (s_0,\ldots,s_n) / I_n (s_0,\ldots,s_n)$

We need to to compute $E_0(M_n (s_0,\ldots,s_n))$, the expectation at time i=0 of $M_n (s_0,\ldots,s_n)$

4. The payout payment function whose expectation provides an estimate of the likelihood of ending with no inventory If $I_n = 0$, $L_n (s_0,\ldots,s_n) = 1$, else $L_n (s_0,\ldots,s_n) = 0$

We need to to compute $E_0(L_n (s_0,\ldots,s_n))$, the expectation at time i=0 of $L_n (s_0,\ldots,s_n)$

5. The payout payment function whose expectation provides an estimate of the profit or loss expected when one ends up with no inventory If $I_n = 0$, $LC_n (s_0,\ldots,s_n) = C_n (s_0,\ldots,s_n)$, else $LC_n (s_0,\ldots,s_n) = 0$.

Here one obtains the estimate as the quotient $E_0(LC_n (s_0,\ldots,s_n))/E_0(L_n (s_0,\ldots,s_n))$

The answer to any of the five questions that are the purpose of this article is a function of the inventory of assets held and the cost of any such inventory held after all unit have been transacted. The inventory and its cost possibly change incrementally as the number of unit assets transacted incrementally increase.

The Difficulties Associated with Traditional Derivatives Pricing

There are three standard methods generally used for derivatives pricing problems: Trees, Monte Carlo and PDE methods

Trinomial Tree

The problem indeed is a trinomial tree type problem. This immediate instinctive approach that comes first to the mind of any trained probabilist is to equivalent to obtaining each one of the 5 estimates by computing the weighted sum of the outcomes in each of the 3^n scenarios by the probability that said scenario would occur.

Using standard tree analysis for the problems at hand, the standard technical term is to conclude that this is a non-recombining tree whose complexity (3^n nodes) explodes with n and therefore is impractical for large values of n.

Monte Carlo Method

The plain Monte Carlo integration algorithm bounds on the approximation are not tight and are provided only in order of magnitude terms and their convergence is known to be slow. As a result, in practice Low-discrepancy sequences are used. However, without extensive empirical investigation, it is easy to see how Low-discrepancy sequences do not compare. Not only do they yield approximate rather than exact results, the best performing sequences (Sobol, Niederreiter) still have an error bounding discrepancy that is an exponential function of n.

PDEs

Trying PDEs is useless here since it is premised on a continuous space and time paradigm that would at best yield only approximate results on a problem that is discrete by definition.

The BICs Based Approach to Computing Multi-period Expectations

The terminology used here is explained in the introductory article on BICs[xiii] .

BICs, BICs set and BICs format used in this example

For each i-th incremental asset unit, there are only three possible next states, with their given probabilities, so it is most natural and efficient to take the BICs format used in this example is as the Arrow Debreu BICs set format and each time t_i i = 0,...,n-1, and the three possible states translate into the following simplified for purose definition of the three BICs:

- An agreement contracted at time t_0 to pay at time t_i $N(s_0,s_1,...,s_i)Prob(s_{i+1}=s_i+1/s_0,s_1,...,s_i)$ units of base currency and to receive $N(s_0,s_1,...,s_i)$ units of base currency if and only if si+1=1

- An agreement contracted at time t_0 to pay at time t_i $N(s_0,s_1,...,s_i)Prob(s_{i+1}=s_i/s_0,s_1,...,s_i)$ units of base currency and to receive $N(s_0,s_1,...,s_i)$ units of base currency if and only if si+1=0

- An agreement contracted at time t_0 to pay at time t_i $N(s_0,s_1,...,s_i)Prob(s_{i+1}=s_i-1/s_0,s_1,...,s_i)$ units of base currency and to receive $N(s_0,s_1,...,s_i)$ units of base currency if and only if s_{i+1}=-1

where we note $Prob(s_{i+1}=s/s_0,s_1,...,s_i)$ as the probability estimated at time t0 that s_{i+1} will be equal to s when the values as yet unknown of $s_0,s_1,...,s_i$ are given and therefore the probability is expressed as a numerical function of those variables. We further note that:

1. The unit price process being a markov chain means unit BICs prices are merely functions of the underlying state and do not depend on the history of all the previous states, i.e. $Prob(s_{i+1}=s/s_0,s_1,...,s_i)=Prob(s_{i+1}=s/s_i)$. This is the assumption that is used in almost all derivatives models ; In a few cases, such as for interest rate derivatives modeling, one takes into account the history of realizations in a in a richer way that can be summarized in a few more variables, usually just one.
2. The notional $N(s_0,s_1,...,s_i)$ of the BICs may however a priori still depend only on the history of all the previous states, leaving us still with a serious computational problem that grows exponentially with n.

Eliminating the Curse of Dimensionality

One of the most important computational contributions of BICs, as will be seen in this example is that by reformulating the problems in BICs terms, it quickly appears that one can achieve powerful ***lossless compression*** in the representation of functional notionals or BICs prices (i.e conditional probabilities here) in a manner that eliminates and transforms the exponential growth in computational cost into a low order polynomial cost computational cost growth.

This method of reducing computational cost is not obvious in other approaches without a BICs framed mind. For instance, make a Google search of the term "non recombining trees" and check the inefficient intellectual contortions made by the authors.

Efficient Representation of Notional Amounts Functions and Breaking Down the Curse of Dimensionality - States Variables Computation

One of the first things one can notice in this effort is that all the payouts depend on the final inventory of assets held and the cost of acquiring such an inventory. Since the probabilities depend solely on the state, we will show that it is sufficient to compress the information contained into $s_0, s_1, \ldots, s_i$ into the information contained into the triplet (I_i C_i,s_i). The difficulty now becomes mapping the possible range of values of ($s_0, s_1, \ldots, s_i$) which contains 3^i elements into the possible range of values (I_i C_i,s_i). Indeed a substantially smaller range of values for (I_i C_i,s_i) will translate into a computational reduction by as much.

$$I_{i+1}(s_0,\ldots,s_{i+1}) = (I_i(s_0,\ldots,s_i)-e_{i+1})^+;$$

$$\text{If } I_i(s_0,\ldots,s_i)>0,\ C_{i+1}(s_0,\ldots,s_{i+1}) = C_i(s_0,\ldots,s_i)-S_{i+1}*(s_{i+1}-s_i),$$

$$\text{else } C_{i+1}(s_0,\ldots,s_{i+1}) = C_i(s_0,\ldots,s_i)-S_{i+1}(s_i)^+ ;$$

There are usually two systematic ways of completing this mapping exercise.

States Variables Computation

The compressed state variables can usually be computated mechanically through a ***forward iterative loop***.

Let's suppose for a given i=0,...,n-1 the set (I_i C_i,s_i) is given. We compute $(I_{i+1}\ C_{i+1},\ s_{i+1})$ as follows:

- For any given (x,y,z) in (I_i,C_i,s_i), (x,y,z) can generate up to 3 distinct (x',y',z') in $(I_{i+1},\ C_{i+1},\ s_{i+1})$ defined by the relations:

$$x' = (x-e)^+;$$

$$y' = y+(S_0+z'*d)*((-e)1_{\{x>0\}}+(-e)^+1_{\{x<=0\}});$$

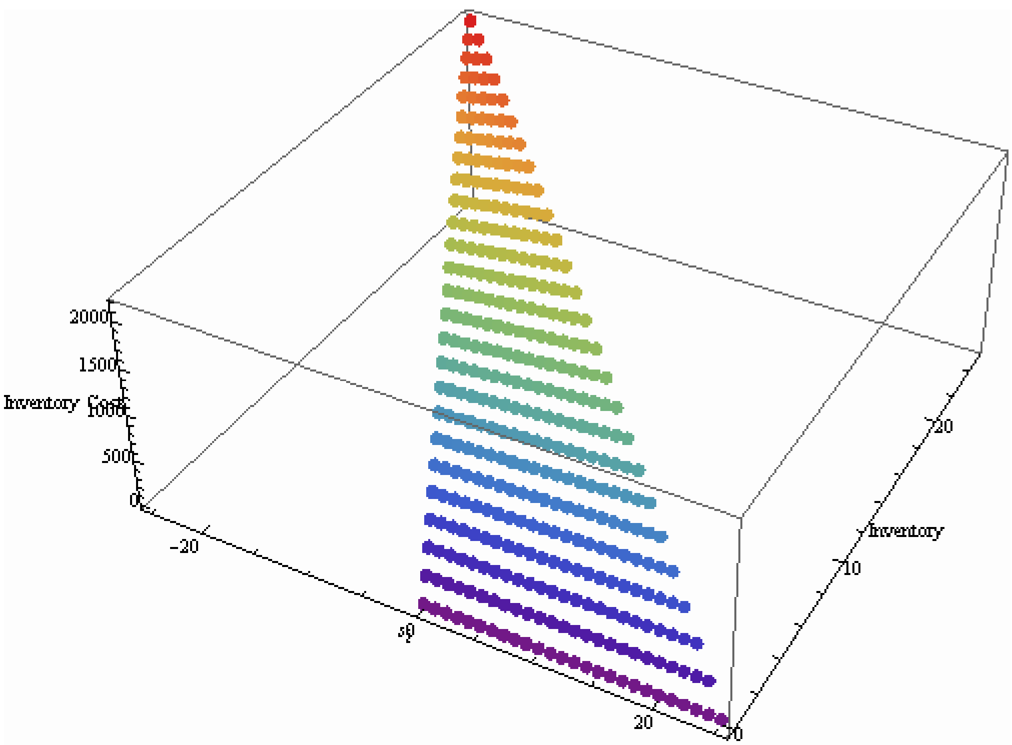
$$z' = z+e;\ \text{where e spans } \{-1,0,1\};$$

- $(I_{i+1}, C_{i+1}, s_{i+1})$ is iteratively built by spanning the elements (x,y,z) in (I_i,C_i,s_i) and e in {-1,0,1} and adding the corresponding element (x',y',z') into the set $(I_{i+1},C_{i+1},s_{i+1})$ provided such an element is not redundant.

The step of computing and checking whether an element (x',y',z') that is already in $(I_{i+1},C_{i+1},s_{i+1})$ is a redundant step that can usually be eliminated with further analysis; similarly, the output of the iterative process at step n can be computed directly without first computing the intermediary steps. It is intellectually a bit more demanding but saves computation time. The Mathematica algorithm that implements this method further details this approach.

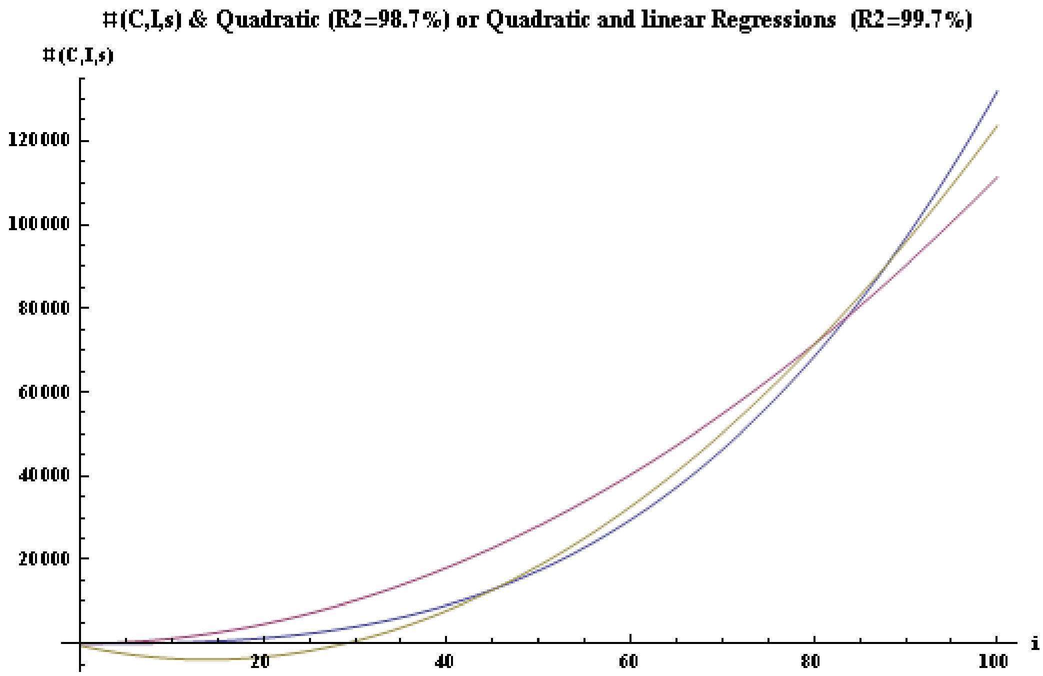
The following graph shows a 3D representation of the triplet (I,C,s) for i=0,..25

Figure 9



It shows an almost one to one correspondence between inventory (width)and inventory cost (height)

Figure 10



Note that R^2 above is a symbol for the coefficient of determination of a linear regression.

Table 2

i	0	10	20	30	40	50	60	70	80	90	100
$\#(C_i, I_i, s_i)$	1	207	1301	4032	9151	17 407	29 551	46 332	68 501	96 807	132 001

Both approaches and possibly other approaches can be admissible if the pricing process is to be repeated over time since once computed, the values can be pre-stored and just called the next time they are needed in the pricing algorithm as will be seen in the next section.

Note that the choice of basis points units spacing is often quite important in helping minimize the number of elements of the state variables. In this particular case, an uniform spacing that is in keeping with practical reality turns out to be computationally optimal. This type of strategic sampling issues, while not obvious, often occurs in numerical analysis. Aside from low discrepancy deterministic sequence which help in large dimension numerical integration approximate computations cases, in exact computations cases, the reader familiar with how a Vandermonde matrix inversion is simplified into a Fast Fourier Transform can appreciate the importance of such issues.

The BICs Backward Replicative Process for Expectations Computations.

Translating and simplifying the BICs Derivatives Static Replication Strategy[xiv] to the expectations computation problem at hand, we obtain a backward iterative sequence of expectation formulas starting with the known terminal payout and which ultimately yields the result sought.

We note Expect(I, C, s, i) the expectation for the vector of the five payout sought after trading the i-th asset given an inventory held I and a cost of acquisition C and with $s_i = s$.

- We know that Expect $(I,C,s,n) = \{I,C, If(I=0,0,C/I),1_{\{I=0\}}, C*1_{\{I=0\}}\}$

- The backward iterative sequence of expectation formulas is given as:

For i= n-1 to 0 step -1,

For j=1 to #StateVariables(i),

(I,C,s) = StateVariables(i)(j);

$Expect(I,C,s,i) = Expect((I -1)^+,C-(S_{0+}(s+1)d))1_{\{I>0\}},s+1,i+1)*Prob(s_{i+1}=s+1/s_0,s_1,...,s_i) + Expect(I,C,s,i+1)*Prob(s_{i+1}=s/s_0,s_1,...,s_i) + Expect(I+1,C+(S_0+(s-1)d), s-1,i+1)*Prob(s_{i+1}=s -1/s_0,s_1,...,s_i);$

- The final element is computed as Expect(0, 0, 0, 0) which is what we were seeking.

Numerical Analysis

We consider the case where we have one original illiquid asset S that we must dispose of. Its original value S_0

equals 100. The unit basis point d = 1. The up, down or same scenarios have probabilities as shown in the graph above**.** The table below show the answer to our original five questions where we consider the unit asset broken down into 10, 15, 20 and 25 and 30 units.

Table 3

Number of Subdivisions	Expected Final Inventory	Expected Final Inventory Fraction	Expected Cost of Inventory	Expected Cost per Unit	Likelihood of No Inventory	Likely Amount Spent if No Inventory
10	5.40068	54.0068 %	520.698	96.28	0.361964 %	-3.20848
15	7.99274	53.2849 %	760.087	95.3159	0.0472256 %	-4.9582
20	10.5846	52.923 %	992.735	94.0739	0.00686011 %	-6.71867
25	13.1766	52.7062 %	1218.67	92.7916	0.00106007 %	-8.48405
30	15.7686	52.562 %	1437.9	91.5012	0.000170494 %	-0.0000174791

One can notice that the expected final inventory fraction quickly converges towards the low end of the buy probability end for the market maker which here is 50% when the underlying asset reaches its lowest possible incremental unit value.

The unit asset cost decreases from the initial 100 to 96.28 and down to 91.5 and is likely to continue to go even lower as granularity increases.

The likelihood of holding no inventory at the end of the process decreases very quickly to zero.

Note that when there is no inventory, the market maker ends up with a positive cash flow *(hence the negative sign)* which represent the bid/offer spread from market making activities.

Conclusion

The liquidity re-establishment process through market making of granular assets seems to be structurally skewed toward efficiently restoring liquidity by purchasing illiquid assets at a discount. It also appears to restore liquidity by acquiring an inventory that is at the low end of the range that could be inferred from inputs of likelihoods of third parties stepping in to bid on the assets.

By contrast, the PPIP prepared by the US Treasury this week is structurally skewed towards efficiently restoring liquidity by purchasing illiquid assets at a premium. This seems to further reinforce the argument of using market making of granular assets as a desirable means of restoring liquidity in a market at minimal costs.

It can be the blue print of lightweight surgical approaches that can be used to restore liquidity in markets that governments consider vital for economic activity and that can be swiftly activated as needed as part for instance of central banks regular open market operations.

The analysis made here assumes the banks would be willing to sell the assets if there was a market for it. However, many banks have a competing interest not to sell because that might reveal deep holes in their balance sheet. Countering such impulses might be addressed through accounting rules changes that dis-incentivizes such behavior or helping set up competing financial institutions with clean balance sheets.

We did not discuss nationalizations in this article as a solution here because one can argue that many of the largest financial institutions are already nationalized (Citi, AIG) and further the Resolution Trust Corporation model used to address the S&L crisis in the late 1980s ended up costing over $160 billions[xv] while the likelihood of such cost borne by the taxpayer in a market making approach seems more remote.

References

[i] an introduction to BICs, see **http://tinyurl.com/cyxhpa**

[ii] So named after US Treasury Secretary Timothy Geithner

[iii] http://www.treas.gov/press/releases/reports/ppip_fact_sheet.pdf

[iv] See Charle Rose interview at http://www.charlierose.com on October 1st, 2008.

[v] See: http://www.risknews.net/public/showPage.html?page=855083

[vi] See Nemo Publius blog at *https://self-evident.org*

[vii] Krugman NY Times Blog on 03/23/09 http://krugman.blogs.nytimes.com/2009/03/23/geithner-plan-arithmetic/

[viii]The additional constraints on σ and E are:

$$\left(E=\frac{S_0^2+S_hS_l}{S_h+S_l}\ \&\&\ \sigma=0\right)\|\left(E=S_h\ \&\&\ \sigma=\sqrt{-S_0^2-S_hS_l+E\left(S_h+S_l\right)}\right)$$
$$\|\left(S_0<E<S_h\ \&\&\ \sqrt{\left(S_h-S_0\right)\left(S_0-E\right)}\le\sigma\le\sqrt{-S_0^2-S_hS_l+E\left(S_h+S_l\right)}\right)$$
$$\|\left(\frac{S_0^2+S_hS_l}{S_h+S_l}<E<S_0\ \&\&\ 0\le\sigma\le\sqrt{-S_0^2-S_hS_l+E\left(S_h+S_l\right)}\right)$$

[ix] See: http://www.risknews.net/public/showPage.html?page=855083

[xi] For an introduction to BICs, see **http://tinyurl.com/cyxhpa**

[xii] See for example BICs 4 Derivatives Volume I : Theory , Chapter VII

[xiii] Introduction to BICs available online at: **http://tinyurl.com/cyxhpa**

[xiv] Again see: **http://tinyurl.com/cyxhpa**

[xv] "Financial Audit: Resolution Trust Corporation's 1995 and 1994 Financial Statements" (PDF). U.S. General Accounting Office. July 1996. pp. 8,13. http://www.gao.gov/archive/1996/ai96123.pdf.