Crash-NIG copula model: regime-switching credit portfolio modeling through the crisis

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Abstract

It is well known that the one-factor copula models are very useful for risk management and measurement applications involving the generation of scenarios for the complete universe of risk factors and the inclusion of CDO structures in a portfolio context. For this objective, it is necessary to have a simple and fast model that is also consistent with the scenario simulation framework. In this paper we present three extensions of the NIG copula model that make the model well defined and powerful for scenario simulation. The Crash-NIG copula model allows for three important features which jointly have not been considered so far: (i) tranches with different maturities modeled in a consistent way, (ii) a portfolio with different rating buckets, relaxing the assumption of a large homogeneous portfolio, and (iii) different correlation regimes. The regime-switching component of the Crash-NIG copula model is especially important in view of the current credit crisis. We also introduce liquidity premiums into the Crash-NIG copula model and show that the actual credit crisis is substantially driven by liquidity effects.

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Introduction

Since researchers (e.g. Li (2000), Finger (2004) and Schoenbucher (2000*a*)) started to apply the Gaussian copula model, originally introduced by Vasicek (1987), for the modeling of a credit portfolio and the pricing of CDO tranches, it became very popular and attracted a lot of attention in financial applications. Especially the one-factor copula approach became very popular. This kind of model sets a restriction on the correlation structure which allows to compute the aggregated portfolio loss much simpler, analytically or semi-analytically: the defaults of different names in the credit portfolio are assumed to be independent conditional on a common market factor. Setting additional restrictions on the model parameters, especially assuming an infinitely large homogeneous portfolio, even allows to compute expected tranche losses for the Gaussian copula model analytically. This made the model immediately to the market standard. However, the model is obviously too simple to describe reality and it is impossible to fit the quotes of different tranches with the same correlation parameter. Thus practitioners started to use the model in the way the Black-Scholes model is used for pricing equity options and so the notion of the correlation smile or skew arose.

Andersen & Sidenius (2005), O'Kane & Schloegl (2003), Hull & White (2004) and Schoenbucher (2000*b*) used different distributions to improve the Gaussian copula model. Andersen & Sidenius (2005), Hull et al. (2005) and Trinh et al. (2005) introduced additional stochastic factors for the same reason. As the main finding of this research was the improvement in the fitting ability by introducing a factor copula with heavy-tailed distributions, Kalemanova et al. (2007) introduced the one-factor NIG (Normal-Inverse Gaussian) copula model. The model found a bright interest among academics, that responded with further extensions of the model or by incorporating other distributions from the same family, as well as among practitioners implementing the model for their applications. Related distributions considered by other researchers after the NIG copula model was introduced include, e.g., variance gamma distributions (see Moosbrucker (2005)), generalized hyperbolic skew Student's t-distribution (see Aas & Haff (2006)) and generalized hyperbolic distribution (see Brunlid (2006)). Albrecher et al. (2007) and Brunlid (2006) generalized the model as Lévy one-factor model.

After the correlation smile was improved by using other distributions, the missing term structure still is a disadvantage of the model. Both, the Gaussian and the NIG models (as well as all analogue models with different distribution assumptions) just average the correlations and other model parameters over the complete lifetime of a CDO tranche. Thus, applying the model to the long-dated tranches is not consistent with the short-dated ones. The practitioners tried to fix this problem by extending the method of base correlations. In contrast to the Gaussian model, the NIG model allows for an extension into the term-structure dimension. This extension is not only helpful for the pricing of CDO tranches with different maturities, but also important for defining a consistent simulation framework for risk management applications. So, the model factors can be defined as stochastic processes and discretized in an arbitrary frequency for a simulation.

The models mentioned before attempt to describe all tranches and maturities of a CDO with only one correlation parameter assuming that the portfolio is homogeneous. Already for one point in time, this assumption is quite strong. In the iTraxx example, there are at least 15 market quotes on one trading day, and it is very ambitious to argue that they all can be explained by only one parameter in the case of the Gaussian model or by two parameters in the case of the NIG model. When using the factor copula model in a simulation framework for risk management, it is difficult to work with a homogeneous portfolio. Using a portfolio consisting of different rating buckets is much more intuitive and allows to take the rating migrations into account while modeling an "average" portfolio credit spread is rather unusual. To circumvent this problem, Desclee et al. (2006) showed that the Gaussian copula model with the LHP (large homogeneous portfolio) setting can be extended to the LHC (large homogeneous cells) setting, which allows to have different correlations for different ratings.

This paper is organized as follows. In the first section we present the term-structure extension of the NIG copula model. In the second section we apply the LHC setting to it. We also show the calibration results of this extended NIG model for a set of quotes and compare them to the LHC Gaussian model. In section three, a regime-switching extension of the NIG model is presented which is especially highly topical in view of the ongoing sub-prime crisis. Although the extension is less important for a stand-alone pricing application, it represents a very important feature for simulation purposes and risk management. A calibration of this model using historical time series which include the sub-prime crisis period until May, 2008, is performed in section four. The results turn out to be very promising and rational, and the calibration ability of the model is rather unproblematic.

1 Term structure extension

The Gaussian factor copula model as well as the NIG factor copula model assume the distributions of the factors to be the same over arbitrary time horizons up to maturity of the tranches. In particular, for pricing CDO tranches the loss distributions for each payment date (i.e. quarterly for iTraxx tranches) are needed for valuation. Thus, applying the model to the long-dated tranches is not consistent with the short-dated ones.

Since the Gaussian distribution is completely determined by two moments, the mean and the standard deviation, it is not possible for the Gaussian factor copula to have different factor distributions for different time horizons. To see this, note that the main idea of the factor models is to describe the returns of the factors $X_i(t)$ and $M(t)$ using stochastic processes with independent increments, zero mean and variance t. In case of the factor Gaussian model, these stochastic processes for the both factors would be simply (uncorrelated) Wiener processes. Then, the asset return defined as $A_i(t) = aM(t) + \sqrt{1-a^2}X_i(t)$ is obviously a Wiener process as well. Normalizing the processes

$$\widetilde{X}_i(t) = \frac{X_i(t)}{\sqrt{t}},\ \widetilde{M}(t) = \frac{M(t)}{\sqrt{t}},\ \widetilde{A}(t) = \frac{A(t)}{\sqrt{t}}$$

we come to the one-factor Gaussian copula model. The normalized factors follow standard normal distributions for any time horizon t. Note that the normalized and non-normalized versions of the factor model are equivalent and lead to identical portfolio loss distributions.

The appropriate process for the factors with NIG-distributed increments is given by $\mathrm{N}_{(s)}(t)$ with a scaling factor s and independent increments $d\mathrm{N}_{(s)}(t) \sim \mathcal{NIG}\left(s\alpha,\, s\beta,\, -s\frac{\beta\gamma^2}{\alpha^2}dt,\, s\frac{\gamma^3}{\alpha^2}dt\right)$, $\gamma = \sqrt{\alpha^2 - \beta^2}$, and has the following properties:

(i) the increments $d\mathrm{N}_{(s)}(t)$ have zero mean and variance dt;

(ii) the process $\mathrm{N}_{(s)}(t)$ has zero mean, variance t, skewness $3\frac{\beta}{s\gamma^2\sqrt{t}}$ and kurtosis $3+3\left(1+4\left(\frac{\beta}{\alpha}\right)^2\right)\frac{\alpha^2}{s^2\gamma^4 t}$;

(iii) $\mathrm{N}_{(s)}(t) \sim \mathcal{NIG}\left(s\alpha,\, s\beta,\, -s\frac{\beta\gamma^2}{\alpha^2}t,\, s\frac{\gamma^3}{\alpha^2}t\right)$.

Thus, we can define the processes of the common (non-normalized) market factor M and of the idiosyncratic (non-normalized) factor X_i as

$$X_i(t) \;=\; \mathrm{N}_{\left(\frac{\sqrt{1-a^2}}{a}\right)}(t),\; M(t) = \mathrm{N}_{(1)}(t)$$

with independent processes X_i and M. Due to the scaling and convolution properties of the NIG distribution (see, e.g. Kalemanova et al. (2007)), the asset return processes $A_i(t) = aM(t) + \sqrt{1-a^2}X_i(t)$ are also processes of the same kind, namely $\mathrm{N}_{(\frac{1}{a})}(t)$. Normalizing the processes

$$\widetilde{X}_i(t) = \frac{X_i(t)}{\sqrt{t}},\; \widetilde{M}(t) = \frac{M(t)}{\sqrt{t}},\; \widetilde{A}(t) = \frac{A(t)}{\sqrt{t}} \tag{1}$$

we do not loose the time dependence in the distribution as it is the case for the Gaussian distribution. For example, the distribution of the normalized market factor would be:

$$\widetilde{M}(t) \;\sim\; \mathcal{NIG}\left(\alpha\sqrt{t},\, \beta\sqrt{t},\, -\frac{\beta\gamma^2}{\alpha^2}\sqrt{t},\, \frac{\gamma^3}{\alpha^2}\sqrt{t}\right)$$

with a zero mean, variance 1, skewness $3\frac{\beta}{\gamma^2\sqrt{t}}$ and kurtosis $3+3\left(1+4\left(\frac{\beta}{\alpha}\right)^2\right)\frac{\alpha^2}{\gamma^4 t}$.

In contrast to the Gaussian distribution with zero skewness and kurtosis 3, the time component now influences the skewness and kurtosis of the NIG distribution. Note that the skewness converges to zero and the kurtosis to 3 with infinitely large t. According to the central limit theorem the sum of a large number of independent returns is approximately normally distributed.

We are going to use the non-standardized version of the NIG copula model with term structure since the both versions are equivalent for pricing and the non-standardized one is more natural and useful for simulations.

Definition 1.1 (Term-structure one-factor NIG copula model)**.** *The asset return up to time t of the i-th issuer in the portfolio, $A_i(t)$, is assumed to be of the form:*

$$A_i(t) = aM(t) + \sqrt{1-a^2}X_i(t), \tag{2}$$

where $M(t), X_i(t), i = 1, ..., m$ are independent processes with $X_i(t) = \mathrm{N}_{\left(\frac{\sqrt{1-a^2}}{a}\right)}(t)$, $M(t) = \mathrm{N}_{(1)}(t)$. Then, $A_i(t) = \mathrm{N}_{(\frac{1}{a})}(t)$.

To further shorten the notations, we will denote:

$$F_{\mathrm{N}_{(s)}(t)}(x) = F_{\mathcal{NIG}}\left(x; s\alpha,\, s\beta,\, -s\frac{\beta\gamma^2}{\alpha^2}t,\, s\frac{\gamma^3}{\alpha^2}t\right). \tag{3}$$

Lemma 1.2. *The distribution function of the portfolio loss at time t is given by*

$$F_{\infty}(t,x) = 1 - F_{\mathrm{N}_{(1)}(t)}\left(\frac{F^{-1}_{\mathrm{N}_{\left(\frac{1}{a}\right)}(t)}(Q(t)) - \sqrt{1-a^2}F^{-1}_{\mathrm{N}_{\left(\frac{\sqrt{1-a^2}}{a}\right)}(t)}(x)}{a}\right), \tag{4}$$

where $Q(t) = \mathbb{Q}[\tau_i \leq t]$ is the risk-neutral probability of default until time t assumed to be identical for all firms in portfolio.

Proof. The proof is analogue to that for the NIG copula model without the time structure component. See e.g. Kalemanova et al. (2007). □

2 Large Homogeneous Cell extension

The LHC framework for modeling credit portfolios was derived by Desclee et al. (2006) for the Gaussian copula model. This article presents a framework for modeling the dynamic behavior of CDO tranches based on a Monte Carlo simulation of the rating migrations and credit spreads as well as the re-pricing of the CDO tranches with the large homogeneous cell (LHC) Vasicek model. The LHC idea can be used not only for rating cells but also for a more detailed classification, e.g. sectors and/or countries. However, one should take care of the portfolio containing many enough issuers to ensure that the assumption of cells with infinitely large number of issuers can be applied. For the case of the iTraxx portfolio containing only 125 issuers three to five rating cells cannot be considered as large enough to be fairly approximated with an infinitely large portfolio cell. However, we accept this drawback intentionally and assume the rating cells of iTraxx to be large enough.

We are going to apply the LHC extension to the term-structure NIG copula model. It is assumed that the portfolio consists of J sub-portfolios, called cells. Each cell $j = 1, ..., J$ contains a sufficiently large number of issuers having the same characteristics:

- the same weight of all issuers in one cell
- the same default probability $Q_j(t)$
- the same recovery R_j
- the same correlation to the market factor $a_j \in (0, 1)$.

The weight of the cell j in the portfolio is denoted as w_j, so that

$$\sum_{j=1}^{J} w_j = 1.$$

We also assume that the recovery rates are the same for all rating cells:

$$R_j = R, j = 1, ..., J.$$

The term-structure LHP NIG model is applied within each cell. The asset return up to time t of the i-th issuer in cell j, $A_{ij}(t)$, is thus assumed to be of the form:

$$A_{ij}(t) = a_j M(t) + \sqrt{1-a_j^2} X_{ij}(t), \tag{5}$$

where $M(t)$ and $X_{ij}(t), i = 1, ..., m_j$ are independent processes such that $X_{ij}(t) = \mathrm{N}_{\left(\frac{\sqrt{1-a_j^2}}{a_j}\right)}(t)$, $M(t) = \mathrm{N}_{(1)}(t)$. Then, $A_{ij}(t) = \mathrm{N}_{(\frac{1}{a_j})}(t)$.

The probability that any instrument from cell j defaults up to time t, conditional on the factor $M(t)$, is

$$p_j(t|M) = F_{\mathrm{N}_{\left(\frac{\sqrt{1-a_j^2}}{a_j}\right)}(t)} \left(\frac{F^{-1}_{\mathrm{N}_{\left(\frac{1}{a_j}\right)}(t)}(Q_j(t)) - a_j M(t)}{\sqrt{1-a_j^2}} \right). \tag{6}$$

and the portfolio loss, conditional on the realization of the systematic factor M, for a portfolio with infinitely large numbers of issuers in each cell is given by:

$$l_t\left(M(t)\right) = \sum_{j=1}^{J} (1-R)\, w_j p_j(t|M). \tag{7}$$

Lemma 2.1. *The loss distribution of an infinitely large homogeneous cell portfolio with the asset returns following a one-factor term-structure NIG copula model is given by*

$$F_{\infty}^{LHC}(t,x) = 1 - F_{\mathrm{N}_{(1)}(t)}\left(l_t^{-1}(x)\right), \tag{8}$$

with $x \in [0,1]$ denoting the percentage portfolio loss. The inverse function $l_t^{-1}(x)$ must be computed numerically.

Proof. The distribution function of the portfolio loss $L(t)$ is given by

$$F_{\infty}^{LHC}(t,x) = \mathbb{Q}[L(t) \leq x] = \mathbb{Q}\left[l_t(M(t)) \leq x\right], \tag{9}$$

where the portfolio loss conditional on $M(t)$, $l_t(M(t))$, is computed as in Equation (7). Note, that the function $l_t(M(t))$ is strictly monotonic decreasing in $M(t)$. Then, $l_t(M(t)) \leq x$ if and only if $M(t) \geq l_t^{-1}(x)$. So we have for the portfolio loss distribution:

$$F_{\infty}^{LHC}(t,x) = \mathbb{Q}\left[M(t) \geq l_t^{-1}(x)\right] = 1 - F_{\mathrm{N}_{(1)}(t)}\left(l_t^{-1}(x)\right). \tag{10}$$

□

For the NIG model, no semi-analytical expression for the expected tranche loss exists. To compute the inverse loss function $l_t^{-1}(x)$ for each spread payment time t, the generation of a look-up table for the function $l_t(x)$ is the most efficient possibility.

For the empirical comparison of the two LHC models, the Gaussian and the NIG, we have used iTraxx data for the 12th of April 2006. For the rating cells extension, some additional input on the rating composition of the portfolio and the ratings-specific default probabilities is required. The iTraxx portfolio contained 0.8% AAA rated issuers, 10.4% AA rated issuers, 42.4% A rated issuers, and 46.4% BBB rated issuers on this day. Choosing the right data for the rating default probabilities is an important issue in the calibration of the LHC model. Recall, that the rough assumption of an infinitely large number of issuers in each cell was made. If the assumption was eligible, one could use the rating-specific credit-spread data to deduce the default probabilities. In reality, the average credit spread for the rating cells of the iTraxx portfolio deviate from the overall EUR rating spreads quite much. However, using default probabilities

Table 1: Pricing iTraxx tranches with different maturities with the LHC model

	Maturity (years)	5	7	10
	iTraxx spread	32 bp	41 bp	52 bp
	AAA spread	10,19 bp	13,75 bp	17,00 bp
	AA spread	14,51 bp	19,40 bp	24,74 bp
	A spread	24,68 bp	32,95 bp	41,41 bp
	BBB spread	44,68 bp	62,09 bp	68,51 bp
Market	0-3%	23,53%	36,875%	48,75%
	3-6%	62,75 bp	189 bp	475 bp
	6-9%	18 bp	57 bp	124 bp
	9-12%	9,25 bp	26,25 bp	56,5 bp
	12-22%	3,75 bp	7,88 bp	19,5 bp
Gaussian LHC	0-3%	28,85%	53,43%	63,19%
	3-6%	92,02 bp	198,81 bp	445,90 bp
	6-9%	32,70 bp	71,91 bp	133,39 bp
	9-12%	13,74 bp	32,88 bp	65,30 bp
	12-22%	2,76 bp	7,88 bp	18,42 bp
	absolute error $2^{nd} - 5^{th}$ tranches	49,44 bp	30,85 bp	48,37 bp
NIG(1) LHC	0-3%	24,92%	48,19%	56,09%
	3-6%	58,42 bp	202,08 bp	475,00 bp
	6-9%	23,4 bp	53,31 bp	124,00 bp
	9-12%	14,25 bp	27,08 bp	51,93 bp
	12-22%	7,59 bp	12,05 bp	18,87 bp
	absolute error $2^{nd} - 5^{th}$ tranches	18,61 bp	22,27 bp	5,20 bp

that are, e.g., much higher than those of the real portfolio, makes it quite impossible to get a good fit of the model to CDO prices. We have taken the CDS spreads of all issuers in the iTraxx portfolio and computed the average rating spreads out of them. Note, that the weighted sum of those spreads should be close to the iTraxx index spread. These spreads are reported in Table 1. These spreads were then used to bootstrap the default probability curves for each rating. The table also contains the calibration results and absolute errors. The overall absolute error of the Gaussian LHC model is 128,66 bp and of the NIG LHC model 46,09 bp. The model parameters of the Gaussian LHC model are $a_{AAA} = 0,6052$, $a_{AA} = 0,0004$, $a_A = 0,7211$, $a_{BBB} = 0,0005$. The model parameters of the NIG LHC model are $a_{AAA} = 0,4217$, $a_{AA} = 0,5139$, $a_A = 0,4522$, $a_{BBB} = 0,2598$, $\alpha = 0,2269$. We choose to fix the parameter $\beta = 0$ since it was shown by Kalemanova et al. (2007) to be the best choice.

3 Crash-NIG copula model

So far two following drawbacks of the Gaussian copula model could be improved by using the NIG distribution. First, the quotes of different tranches with the same maturity can be fitted quite well with one correlation parameter. Second, the tranches with different maturities can be modeled consistently. Besides, applying the LHC approach gives more free parameters and is more suitable for a simulation risk management application. The correlation parameters were assumed to be constant over time so far. It is no problem for a pricing application since the correlation parameter can be updated every day by a new calibration. However, it is not realistic to assume the correlations being constant for a simulation application. Especially the on going financial crisis has shown an extreme correlation shift. For this reason, we want to integrate the possibility of different correlation regimes to the NIG model. First, we consider two regimes: the first is the regime of an usual correlation and the second of a high (crash) correlation. Thereby, the model has to satisfy some requirements that are important for the simulation framework. In the next proposition, these requirements are listed and the Crash-NIG copula model is derived.

Proposition 3.1. *Consider the Crash-NIG model, which is given by*

$$dA_{ij}(t) = a_j dM(t) + \sqrt{1-a_j^2}\, dX_{ij}(t), \tag{11}$$

with independent factors following NIG distributions $dM(t) = d\mathrm{N}_{(1)}(t)$ *and* $dX_{ij}(t) = d\mathrm{N}_{\left(\frac{\sqrt{1-a_j^2}}{a_j}\right)}(t)$*, i.e.,*

$$\begin{aligned} dM(t) &\sim \mathcal{NIG}\left(\alpha,\, \beta,\, -\frac{\beta\gamma^2}{\alpha^2}dt,\, \frac{\gamma^3}{\alpha^2}dt\right), & (12)\\ dX_{ij}(t) &\sim \mathcal{NIG}\left(\frac{\sqrt{1-a_j^2}}{a_j}\alpha,\, \frac{\sqrt{1-a_j^2}}{a_j}\beta,\, -\frac{\sqrt{1-a_j^2}}{a_j}\frac{\beta\gamma^2}{\alpha^2}dt,\, \frac{\sqrt{1-a_j^2}}{a_j}\frac{\gamma^3}{\alpha^2}dt\right) & (13)\end{aligned}$$

in the first state, and in the second state by

$$d\widehat{A}_{ij}(t) = \widehat{a}_j d\widehat{M}(t)(t) + \sqrt{1-\widehat{a}_j^2}\, d\widehat{X}_{ij}(t), \tag{14}$$

with independent factors $\widehat{M}$ *and* $\widehat{X}_{ij}$ *following a NIG distribution. Let us further assume that Crash-NIG model has to satisfy the following requirements:*

(i) The distributions of both factors in different states are stable under convolution.

(ii) The asset return has the same distribution in both states to ensure an easy derivation of the default thresholds.

(iii) The distributions of the factors in both states have zero mean.

(iv) The distribution of the market factor does not depend on the correlation.

Then, there exists a real number $k > 0$ *such that the asset return in the second state can be written as:*

$$d\widehat{A}_{ij}(t) = a_j d\widehat{M}(t) + \sqrt{1-a_j^2}\, d\widehat{X}_{ij}(t), \tag{15}$$

with the distributions of the factors given by

$$\begin{aligned} d\widehat{M}(t) &\sim \mathcal{NIG}\left(\alpha,\, \beta,\, -k\frac{\beta\gamma^2}{\alpha^2}dt,\, k\frac{\gamma^3}{\alpha^2}dt\right), & (16)\\ d\widehat{X}_{ij}(t) &\sim \mathcal{NIG}\left(\frac{\sqrt{1-a_j^2}}{a_j}\alpha,\, \frac{\sqrt{1-a_j^2}}{a_j}\beta,\right. & (17)\\ &\qquad \left. -\frac{1-ka_j^2}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\beta\gamma^2}{\alpha^2}dt,\, \frac{1-ka_j^2}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\gamma^3}{\alpha^2}dt\right), \end{aligned}$$

and the distribution of $d\widehat{A}_{ij}(t)$ is the same as $dA_{ij}(t)$.

Proof. See Appendix A. $\square$

Recall, that in the first correlation regime, the variance of all factor changes is dt. Now, the variance of the factors in the second regime is given by

$$V_{d\widehat{M}} = kdt, \; V_{d\widehat{X}_i} = \frac{1-ka_j^2}{1-a_j^2}dt.$$

Thus, the correlation of asset returns of an issuer i_1 from the rating cell j_1 and an issuer i_2 from the rating cell j_2 is

$$Corr\left(dA_{i_1j_1}(t), dA_{i_2j_2}(t)\right) = a_{j_1}a_{j_2}V_{d\widehat{M}} = a_{j_1}a_{j_2}kdt.$$

The higher correlation in the second regime is implied by the higher variance of the market factor, i.e. by choosing $k > 1$. The variance of the idiosyncratic factor is then lower than normal.

Based on the results of the last proposition, we can define the regime-switching extension of the NIG copula model.

Definition 3.2 (Crash-NIG copula model)**.** *The asset return of the i-th issuer in cell j for $j = 1, ..., J$, $A_{ij}(t)$, is assumed to be of the form:*

$$dA_{ij}(t) = a_j dM(t) + \sqrt{1-a_j^2}dX_{ij}(t), \tag{18}$$

where $M(t), X_{ij}(t), i = 1, ..., m$ are independent processes with the following distributions:

$$dM(t) \sim \mathcal{NIG}\left(\alpha,\, \beta,\, -\Lambda_t^2\frac{\beta\gamma^2}{\alpha^2}dt,\, \Lambda_t^2\frac{\gamma^3}{\alpha^2}dt\right), \tag{19}$$

$$dX_{ij}(t) \sim \mathcal{NIG}\left(\frac{\sqrt{1-a_j^2}}{a_j}\alpha,\, \frac{\sqrt{1-a_j^2}}{a_j}\beta,\, -\frac{1-\Lambda_t^2a_j^2}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\beta\gamma^2}{\alpha^2}dt,\, \frac{1-\Lambda_t^2a_j^2}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\gamma^3}{\alpha^2}dt\right).$$

Λ_t is a Markov process with state space $\{1, \lambda\}$, an initial distribution $\pi = \{\pi_1, \pi_2\}$ and a (2×2) transition function $\{P(h)\}_{h\geq 0}$. The distribution of the increment of the asset return is $dA_{ij}(t) = \mathrm{N}_{(\frac{1}{a_j})}(t)$, i.e.,

$$dA_{ij}(t) \sim \mathcal{NIG}\left(\frac{1}{a_j}\alpha,\, \frac{1}{a_j}\beta,\, -\frac{1}{a_j}\frac{\beta\gamma^2}{\alpha^2}dt,\, \frac{1}{a_j}\frac{\gamma^3}{\alpha^2}dt\right).$$

The Crash-NIG copula model can be easily extended to a higher number of regimes. Then, the Markov process Λ_t has the state space $\{1, \lambda_1, ..., \lambda_{n-1}\}$, an initial distribution $\pi = \{\pi_1, \pi_2, ..., \pi_n\}$ and a $(n \times n)$ transition function $\{P(h)\}_{h\geq 0}$.

Now we are interested in the distributions of the cumulated returns on $[0, t]$ of the model factors. Since the third and the fourth parameter of the NIG distribution must be summed up by convolution, the following random variables are of special interest: $T^r(t) := (T_1^r(t), T_2^r(t))'$ giving the duration of the stay in state i starting from the state r at time $t = 0$:

$$T_i^r(t) = \int_0^t 1_{\{\text{state i at time s}\}}ds. \tag{20}$$

Using

$$\int_0^t \Lambda_s^2 ds = \int_0^t \left(1_{\{\text{state 1 at time s}\}} + \lambda^2 1_{\{\text{state 2 at time s}\}}\right) ds = T_1^r(t) + \lambda^2 T_2^r(t), \tag{21}$$

we know that the distributions of $M(t)$ and $X_{ij}(t)$, the cumulated returns on $[0,t]$, conditional on the realization of $T^r(t)$, are NIG with the following parameters:

$$M(t)|T^r(t) \sim \mathcal{NIG}\left(\alpha,\, \beta,\, -\left(T_1^r(t) + \lambda^2 T_2^r(t)\right)\frac{\beta\gamma^2}{\alpha^2},\, \left(T_1^r(t) + \lambda^2 T_2^r(t)\right)\frac{\gamma^3}{\alpha^2}\right), \tag{22}$$

$$X_{ij}(t)|T^r(t) \sim \mathcal{NIG}\left(\frac{\sqrt{1-a_j^2}}{a_j}\alpha,\, \frac{\sqrt{1-a_j^2}}{a_j}\beta,\right. \tag{23}$$

$$\left. -\frac{t - a_j^2\left(T_1^r(t) + \lambda^2 T_2^r(t)\right)}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\beta\gamma^2}{\alpha^2},\, \frac{t - a_j^2\left(T_1^r(t) + \lambda^2 T_2^r(t)\right)}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\gamma^3}{\alpha^2}\right).$$

The distribution of $A_{ij}(t)$ is as before $\mathrm{N}_{(\frac{1}{a_j})}(t)$.

If the density function $f_{T^r(t)} : \Omega_t \to \mathbb{R}$ with $\Omega_t = [0,t]^2$ of the duration $T^r(t) := (T_1^r(t), T_2^r(t))'$ of the stay in some state starting from state $r \in \{1,2\}$ is known, the unconditional densities of the factors $M(t)$ and $X_{ij}(t)$ are

$$f_{M(t)}(x) = \int_{\Omega_t} f_{\mathcal{NIG}}\left(x;\alpha,\, \beta,\, -\left(z_1 + \lambda^2 z_2\right)\frac{\beta\gamma^2}{\alpha^2},\, \left(z_1 + \lambda^2 z_2\right)\frac{\gamma^3}{\alpha^2}\right) f_{T^r(t)}(z_1,z_2)d(z_1,z_2)$$

$$\begin{aligned} f_{X_{ij}(t)}(x) = \int_{\Omega_t} f_{\mathcal{NIG}}\Bigg(&x;\frac{\sqrt{1-a_j^2}}{a_j}\alpha,\, \frac{\sqrt{1-a_j^2}}{a_j}\beta,\, -\frac{t-a_j^2\left(z_1+\lambda^2 z_2\right)}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\beta\gamma^2}{\alpha^2}, \\ &\frac{t-a_j^2\left(z_1+\lambda^2 z_2\right)}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\gamma^3}{\alpha^2}\Bigg) f_{T^r(t)}(z_1,z_2)d(z_1,z_2). \end{aligned}$$

Unfortunately, the distributions of the duration of stay $T^r(t)$ are very complicated even for two states. To our knowledge it is impossible to compute the unconditional densities of the factors analytically. A numerical integration would be very time and memory consuming and could only make sense if a very exact pricing on a single day is needed and can be performed on a high-end machine. However, the four moments of the unconditional distributions can be computed quite easily. So, an approximation of the unconditional distributions with a NIG distribution matching the four moments seems to be a good alternative to the exact computation. The next proposition gives the formulas for the four moments.

Proposition 3.3. *Denoting the variance by $\mathbb{V}$, the skewness by $\mathbb{S}$ and the kurtosis by $\mathbb{K}$, the moments of the uncondition distribution of the factor $M(t)$ are:*

$$\begin{aligned}
\mathbb{E}\left(M(t)\right) &= 0\\
\mathbb{V}\left(M(t)\right) &= \mathbb{E}\left(T_1^r(t)+\lambda^2 T_2^r(t)\right)\\
\mathbb{S}\left(M(t)\right) &= \frac{3\beta}{\gamma^2}\mathbb{E}\left(\frac{1}{\sqrt{T_1^r(t)+\lambda^2 T_2^r(t)}}\right)\\
\mathbb{K}\left(M(t)\right) &= 3+3\left(1+4\left(\frac{\beta}{\alpha}\right)^2\right)\mathbb{E}\left(\frac{1}{T_1^r(t)+\lambda^2 T_2^r(t)}\right)\frac{\alpha^2}{\gamma^4}.
\end{aligned}$$

The moments of the uncondition distribution of the factor $X_{ij}(t)$ are:

$$\begin{aligned}
\mathbb{E}\left(X_{ij}(t)\right) &= 0\\
\mathbb{V}\left(X_{ij}(t)\right) &= \frac{t-a_j^2\mathbb{E}\left(T_1^r(t)+\lambda^2 T_2^r(t)\right)}{1-a_j^2}\\
\mathbb{S}\left(X_{ij}(t)\right) &= \frac{3\beta a_j}{\gamma^2}\mathbb{E}\left(\frac{1}{\sqrt{t-a_j^2\left(T_1^r(t)+\lambda^2 T_2^r(t)\right)}}\right)\\
\mathbb{K}\left(X_{ij}(t)\right) &= 3+3\left(1+4\left(\frac{\beta}{\alpha}\right)^2\right)\mathbb{E}\left(\frac{1}{t-a_j^2\left(T_1^r(t)+\lambda^2 T_2^r(t)\right)}\right)\frac{a_j^2\alpha^2}{\gamma^4}.
\end{aligned}$$

Proof. See Appendix B. □

Now we consider the moment matching exemplarily for the market factor. If we want to approximate $M(t)$ with a NIG distribution by moment matching:

$$M(t) \simeq \mathcal{NIG}\left(\widehat{\alpha}(t),\, \widehat{\beta}(t),\, \widehat{\mu}(t),\, \widehat{\delta}(t)\right), \tag{24}$$

then, setting $\widehat{\gamma}(t)=\sqrt{\widehat{\alpha}^2(t)-\widehat{\beta}^2(t)}$, the following system of four equations has to be solved for $\widehat{\alpha}(t)$, $\widehat{\beta}(t)$, $\widehat{\mu}(t)$, $\widehat{\delta}(t)$:

$$\begin{aligned}
\widehat{\mu}(t)+\widehat{\delta}(t)\frac{\widehat{\beta}(t)}{\widehat{\gamma}(t)} &= 0\\
\widehat{\delta}(t)\frac{\widehat{\alpha}^2(t)}{\widehat{\gamma}^3(t)} &= \mathbb{E}\left(T_1^r(t)+\lambda^2 T_2^r(t)\right)\\
\frac{3\widehat{\beta}(t)}{\widehat{\alpha}(t)\sqrt{\widehat{\delta}(t)\widehat{\gamma}(t)}} &= \frac{3\beta}{\gamma^2}\mathbb{E}\left(\frac{1}{\sqrt{T_1^r(t)+\lambda^2 T_2^r(t)}}\right)\\
\left(1+4\left(\frac{\widehat{\beta}(t)}{\widehat{\alpha}(t)}\right)^2\right)\frac{1}{\widehat{\delta}(t)\widehat{\gamma}(t)} &= \left(1+4\left(\frac{\beta}{\alpha}\right)^2\right)\mathbb{E}\left(\frac{1}{T_1^r(t)+\lambda^2 T_2^r(t)}\right)\frac{\alpha^2}{\gamma^4}.
\end{aligned}$$

In general, this system of equations cannot be solved analytically. In the special case of

$\beta = 0$, however, the system is easy to solve. The parameters for $M(t)$ are:

$$\begin{aligned}
\widehat{\beta}(t) &= 0,\ \widehat{\mu}(t) = 0 \text{ and, using } \widehat{\gamma}(t) = \sqrt{\widehat{\alpha}^2(t) - \widehat{\beta}^2(t)} = \widehat{\alpha}(t), \\
\widehat{\alpha}(t) &= \frac{\alpha}{\sqrt{\mathbb{E}\left(T_1^r(t) + \lambda^2 T_2^r(t)\right)\mathbb{E}\left(\frac{1}{T_1^r(t)+\lambda^2 T_2^r(t)}\right)}}, \\
\widehat{\delta}(t) &= \alpha \cdot \sqrt{\frac{\mathbb{E}\left(T_1^r(t) + \lambda^2 T_2^r(t)\right)}{\mathbb{E}\left(\frac{1}{T_1^r(t)+\lambda^2 T_2^r(t)}\right)}}.
\end{aligned}$$

The expectation $\mathbb{E}\left(T_1^r(t) + \lambda^2 T_2^r(t)\right)$ can be easily computed as:

$$\mathbb{E}\left(T_1^r(t) + \lambda^2 T_2^r(t)\right) = \mathbb{E}\left(T_1^r(t)\right) + \lambda^2 \mathbb{E}\left(T_2^r(t)\right) = h_1^r(t) + \lambda^2 h_2^r(t),$$

with

$$\begin{aligned}
h_i^r(t) &= \mathbb{E}\left(T_i^r(t)\right) = \mathbb{E}\left(\int_0^t 1_{\{\text{state i at time s}\}} ds\right) = \int_0^t \mathbb{Q}\left[\text{state i at time s}\right] ds \\
&= \int_0^t \left(exp\left(sO\right)\right)_{r,i} ds
\end{aligned} \tag{25}$$

for $i = 1, 2$, where $O := \underset{h \downarrow 0}{lim} \frac{P(h)-I}{h}$ is the intensity matrix of the transition function $P(h)$, i.e. $P(h) = exp\{hO\} = \sum_{k=0}^{\infty} \frac{h^k}{k!} O^k$ (see, e.g., Anderson (1991) and Mai (2007)).

This shows that the variance of the distributions of $M(t)$ and $X_{ij}(t)$ is easy to compute. The computation for the skewness and kurtosis is not that straightforward. For applications where the computation speed is a more important issue than the accuracy, the simple fitting of the variance of the distributions may be a better choice. In this case, the parameters of the approximating NIG distribution can be chosen as described in the next remark.

Remark 3.4. *Approximation of $M(t)$ and $X_{ij}(t)$ with*

$$\begin{aligned}
M(t) &\simeq \mathcal{NIG}\left(\alpha,\, \beta,\, -\left(h_1^r(t) + \lambda^2 h_2^r(t)\right) \frac{\beta\gamma^2}{\alpha^2},\, \left(h_1^r(t) + \lambda^2 h_2^r(t)\right) \frac{\gamma^3}{\alpha^2}\right) \\
X_{ij}(t) &\simeq \mathcal{NIG}\left(\frac{\sqrt{1-a_j^2}}{a_j}\alpha,\, \frac{\sqrt{1-a_j^2}}{a_j}\beta,\right. \\
&\qquad \left. -\frac{t - a_j^2\left(h_1^r(t) + \lambda^2 h_2^r(t)\right)}{1-a_j^2} \frac{\sqrt{1-a_j^2}}{a_j} \frac{\beta\gamma^2}{\alpha^2},\, \frac{t - a_j^2\left(h_1^r(t) + \lambda^2 h_2^r(t)\right)}{1-a_j^2} \frac{\sqrt{1-a_j^2}}{a_j} \frac{\gamma^3}{\alpha^2}\right)
\end{aligned}$$

fits the first two moments of the exact distributions. The third and the fourth moments of the approximate distribution are not higher than those of the exact distribution. In the special case of a non-skewed distributions, i.e. $\beta = 0$, the skewness, is zero for the approximate and the exact distributions.

Proof. The computation of the first two moments of the approximative distributions is straightforward. They are equal to the first two moments in the Proposition 3.3. The skewness of the approximate distribution of $M(t)$ is

$$\frac{3\beta}{\gamma^2}\left(\frac{1}{\sqrt{\mathbb{E}\left(T_1^r(t)+\lambda^2 T_2^r(t)\right)}}\right) \leq \frac{3\beta}{\gamma^2}\mathbb{E}\left(\frac{1}{\sqrt{T_1^r(t)+\lambda^2 T_2^r(t)}}\right).$$

The inequality is given by Jensen's inequality since $f(x) = \frac{1}{\sqrt{x}}$ is a convex function. The proof for the kurtosis is analogue. □

All the results are analogue for more than two states changing $h_1^r(t) + \lambda^2 h_2^r(t)$ for a suitable expression, e.g. $h_1^r(t) + \lambda_1^2 h_2^r(t) + \lambda_2^2 h_3^r(t)$ for three states.

We have simulated some examples of the model and computed the unconditional distribution functions in order to get a feeling of how good the described approximations are. The first example is a two state model with following parameter:

$$\begin{aligned} O &= \begin{pmatrix} -0.0038 & 0.0038 \\ 0.0120 & -0.0120 \end{pmatrix} \\ \pi &= (0,1),\ \alpha = 0.4,\ \lambda = 2,\ a_j = 0.5,\ t = 3. \end{aligned}$$

The states of the Markov process are simulated over a 3-years period. The durations of the stays corresponding to the simulated paths were used to simulate the unconditional distribution of $M(t)$ and $X_i(t)$. Figure 1 shows the histogram of $T_1^r(t)+\lambda^2 T_2^r(t)$ as well as the unconditional distribution function of the factor $M(t)$. The distributions of the approximation of the factor $M(t)$ fitting two and four moments are also plotted to compare the approximation error. The three distribution functions are nearly the same for a model with two states. The results for the factor $X_{ij}(t)$ are similar. We get even similar results if we increase the transition probability of one state into another.

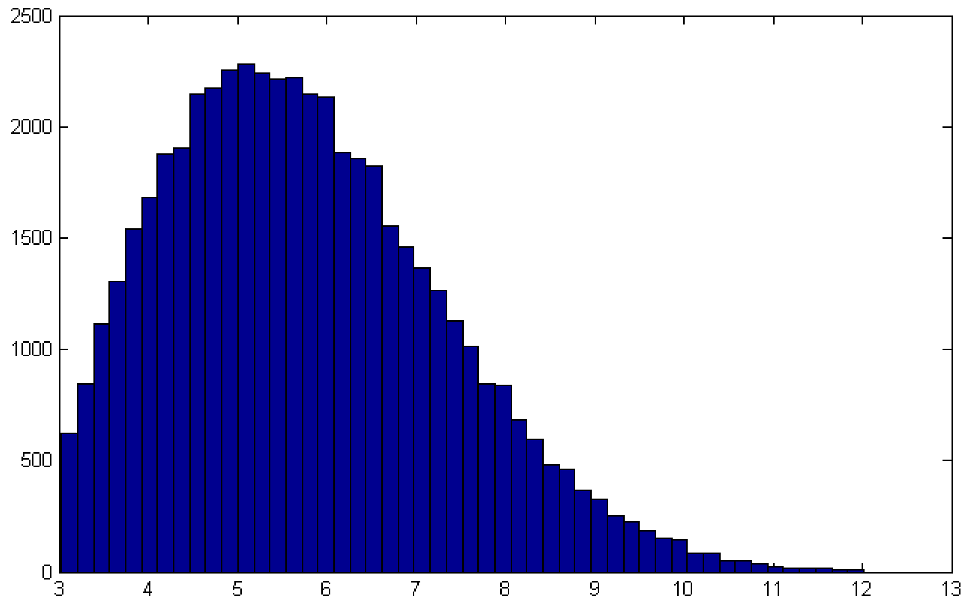
Next, we consider two examples of a three-state model. The first example has the following parameter:

$$\begin{aligned} O &= \begin{pmatrix} -0.0139 & 0.0140 & -0.0001 \\ 0.0142 & -0.0231 & 0.0089 \\ 0.0127 & -0.0001 & -0.0126 \end{pmatrix} \\ \pi &= (0,1,0),\ \alpha = 0.4,\ \lambda_1 = 0.25,\ \lambda_2 = 1.75,\ a_j = 0.5,\ t = 3. \end{aligned}$$

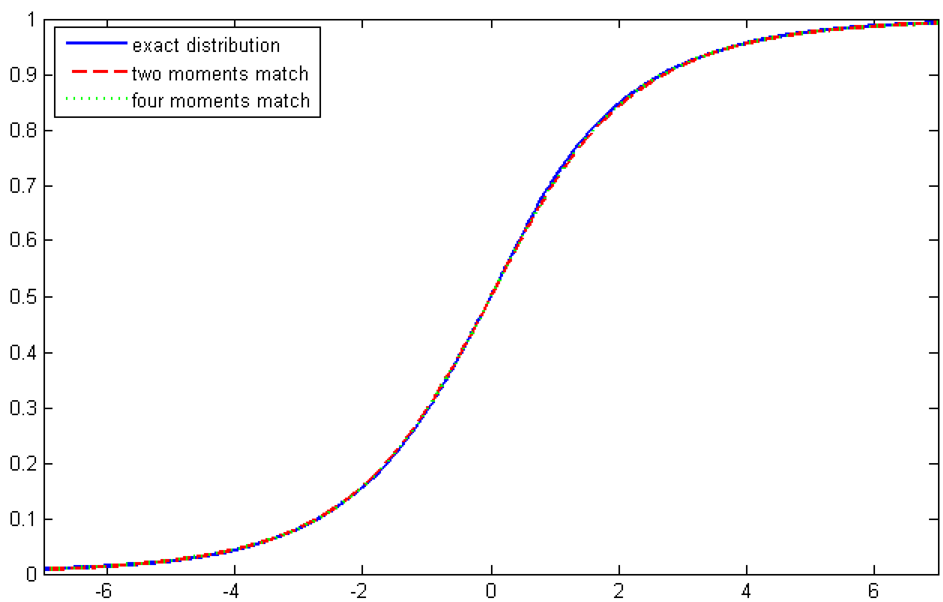
The approximation of the unconditional distribution of the factors with a NIG distribution matching two or four moments is also very exact.

Another three-state model under consideration is a model with an absorbing state: once we are in state three, it is not possible to escape from it. The parameters of the model are:

$$\begin{aligned} O &= \begin{pmatrix} -0.0048 & 0.0048 & 0 \\ 0.0165 & -0.0216 & 0.0051 \\ 0 & 0 & 0 \end{pmatrix} \\ \pi &= (0,1,0),\ \alpha = 0.4,\ \lambda_1 = 0.25,\ \lambda_2 = 1.75,\ a_j = 0.5,\ t = 3. \end{aligned}$$

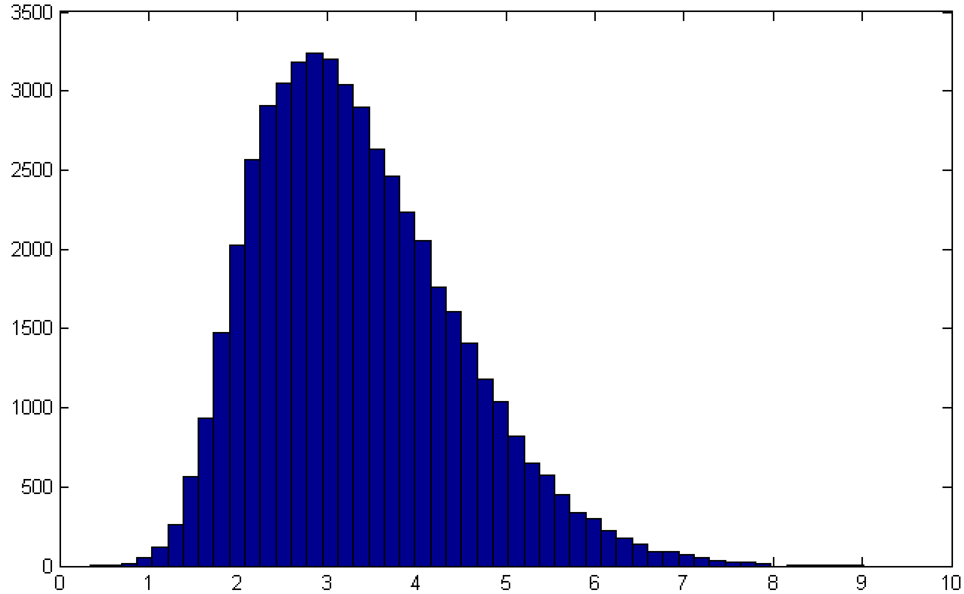


(a) Histogram of $T_1^r(t) + \lambda^2 T_2^r(t)$

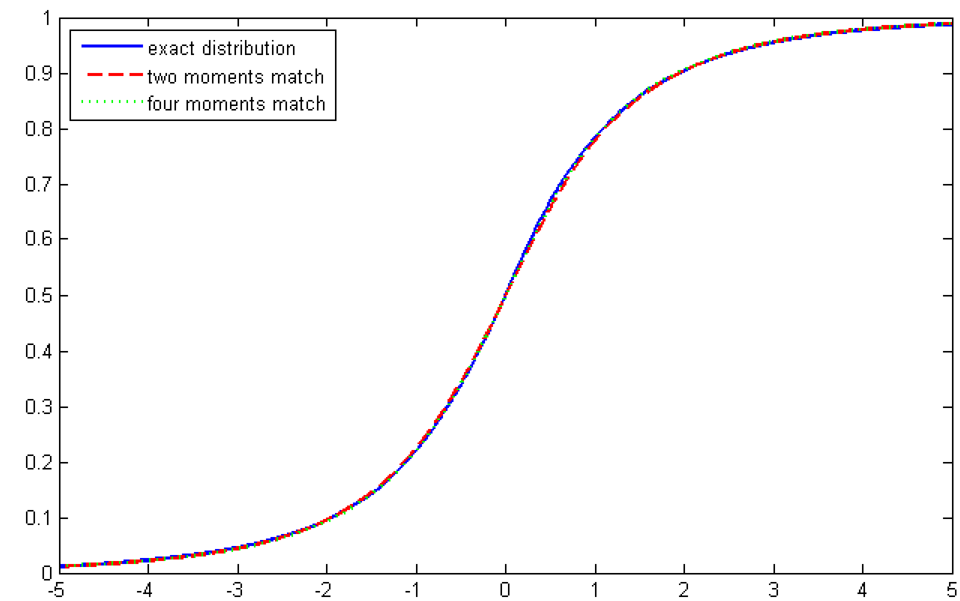


(b) Distribution function of $M(t)$

Figure 1: Example of a two-state model

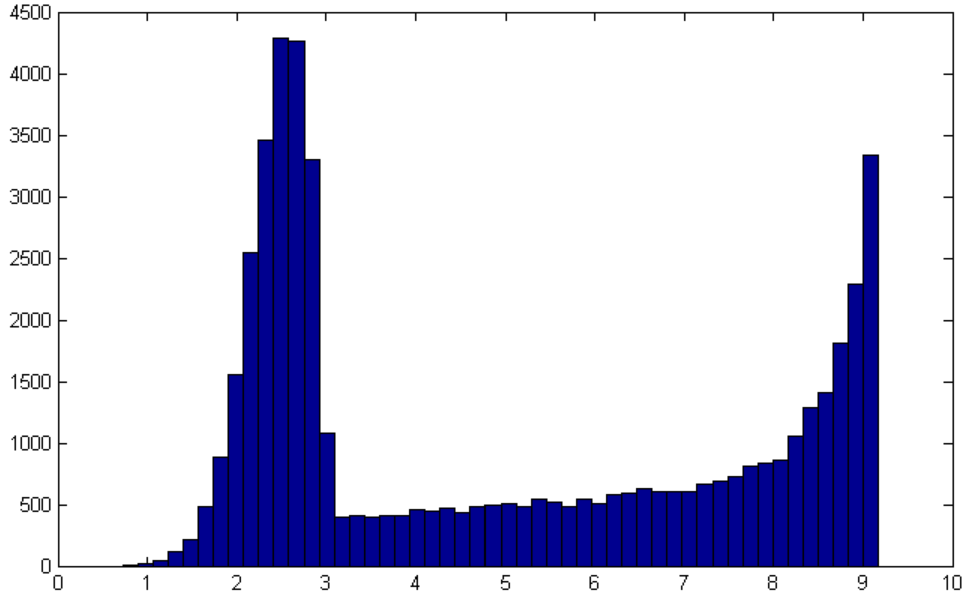


(a) Histogram of $T_1^r(t) + \lambda_1^2 T_2^r(t) + \lambda_2^2 T_3^r(t)$

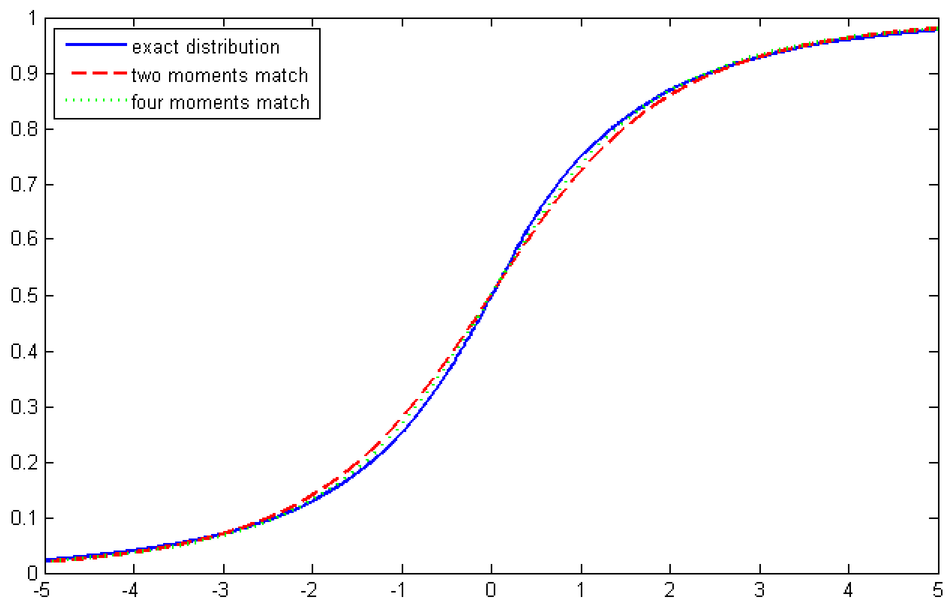


(b) Distribution function of $M(t)$

Figure 2: Example of a three-state model with no absorbing states



(a) Histogram of $T_1^r(t) + \lambda_1^2 T_2^r(t) + \lambda_2^2 T_3^r(t)$



(b) Distribution function of $M(t)$

Figure 3: Example of a three-state model with an absorbing state

For this case, the distribution of $T_1^r(t) + \lambda_1^2 T_2^r(t) + \lambda_2^2 T_3^r(t)$ is quite different than in the previous examples. The approximations of the distribution of factor $M(t)$ are not that exact anymore. However, they are still very accurate. The difference between the exact unconditional distribution function of $X_i(t)$ and the approximations is much smaller.

Now we have all distributions necessary to update the formulas of the distribution of a large homogeneous cell portfolio loss for the Crash-NIG copula model in hand.

Lemma 3.5. *The loss distribution of an infinitely large homogeneous cell portfolio with the asset returns following a Crash-NIG copula model with two states is given by*

$$F_{\infty}^{LHC}(t,x) = 1 - F_{\mathcal{NIG}}\left(l_t^{-1}(x); \alpha,\, \beta,\, -\left(h_1^r(t) + \lambda^2 h_2^r(t)\right)\frac{\beta\gamma^2}{\alpha^2},\, \left(h_1^r(t) + \lambda^2 h_2^r(t)\right)\frac{\gamma^3}{\alpha^2}\right), \quad (26)$$

with $x \in [0,1]$ denoting the percentage portfolio loss. The function $l_t(M(t))$ is the portfolio loss conditional on the realization of the systematic factor $M(t)$ and is given by:

$$\begin{aligned} l_t\left(M(t)\right) \quad = \quad & \sum_{j=1}^{J}(1-R)\, w_j F_{\mathcal{NIG}}\left(\frac{C_j(t) - a_j M(t)}{\sqrt{1-a_j^2}};\, \frac{\sqrt{1-a_j^2}}{a_j}\alpha,\, \frac{\sqrt{1-a_j^2}}{a_j}\beta,\right. \qquad (27) \\ & \left. -\frac{t - a_j^2\left(h_1^r(t) + \lambda^2 h_2^r(t)\right)}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\beta\gamma^2}{\alpha^2},\, \frac{t - a_j^2\left(h_1^r(t) + \lambda^2 h_2^r(t)\right)}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\gamma^3}{\alpha^2}\right). \end{aligned}$$

The default thresholds are computed as

$$C_j(t) = F_{\mathcal{NIG}}^{-1}\left(Q_j(t);\, \frac{1}{a_j}\alpha,\, \frac{1}{a_j}\beta,\, -\frac{1}{a_j}\frac{\beta\gamma^2}{\alpha^2}t,\, \frac{1}{a_j}\frac{\gamma^3}{\alpha^2}t\right).$$

Proof. Proof is analogue to the single-regime LHC NIG model. □

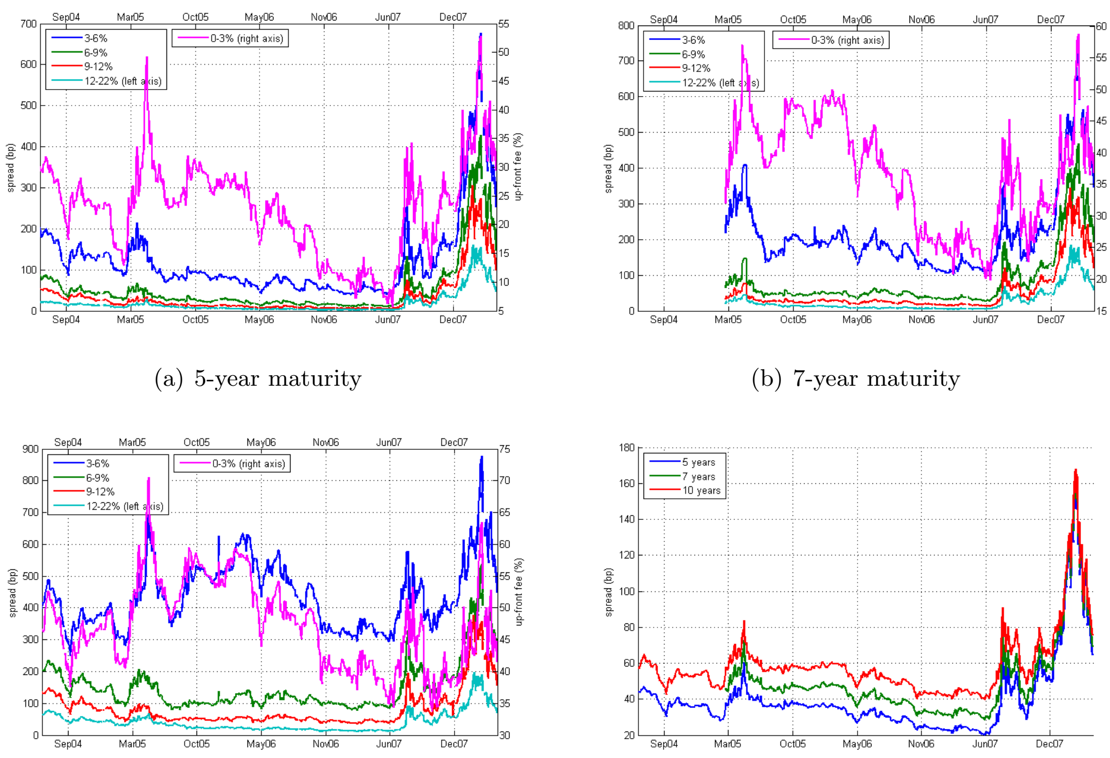
Again, it is straightforward to change the formulas for the model with more than two states by changing the expression $h_1^r(t) + \lambda^2 h_2^r(t)$.

As it was already the case for the single-regime LHC NIG copula model, there exist no analytical expressions for the expected tranche loss. They have to be computed numerically by approximating the corresponding integrals over the portfolio loss distribution function.

We have implemented two versions for CDO valuations using the Crash-NIG copula model. First, the inverse conditional loss function $l_t^{-1}(x)$ was implemented with the help of a look-up table. For all NIG distribution and inverse distribution functions the corresponding routines where called in each evaluation point. Such implementation is quite time consuming taking approximately 25 seconds in Matlab for one trading day. Of course, it is not acceptable for the calibration of a 4 year history and a simulation. So we have implemented a second, vector version with additional look-up tables. Since it is necessary to call the NIG distribution and inverse distribution functions with different parameters and for different time increments for a calibration, this is exactly the part of the pricing function that takes most of the computation time. A possibility to avoid this is the creation of look-up tables for the NIG distribution and the NIG inverse distribution functions. This implementation takes us only 3 seconds to price the iTraxx tranches on one day, and 25 seconds for a simultaneous pricing on 200 days[1]. Now, the n-dimensional interpolation in the look-up tables is taking most of the computation time. We have not spent any additional effort trying to accelerate this and have simply used the Maltab function "interpn". However, we believe that this is not the fastest possible implementation and Matlab in general is not the fastest programming environment.

4 Calibration results of the Crash-NIG copula model

Collecting and preparing the relevant data is a very important and sensitive issue in the calibration of the Crash-NIG copula model. The reasons for this are, first, the complexity of the financial instrument we are going to price, and, second, the application of the more detailed information by the large homogeneous cell model in comparison to the large homogeneous portfolio model. Note that, the LHC model needs the information on the rating cells of the portfolio while, the average default probability deduced from the iTraxx index spread is enough for the LHP model.



(a) 5-year maturity

(b) 7-year maturity

(c) 10-year maturity

(d) iTraxx index spread

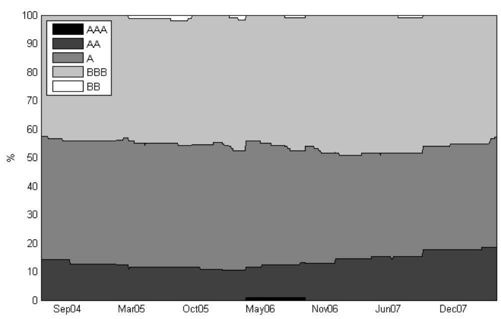
Figure 4: Market quotes of the iTraxx index and tranches

We use the complete history of the iTraxx Europe tranched index since its origination on the 21th of June 2004 until the 6th of May 2008. The 7-year maturity became available only since the end of March 2005. The data is presented in Figure 4. The data on the index spread for the

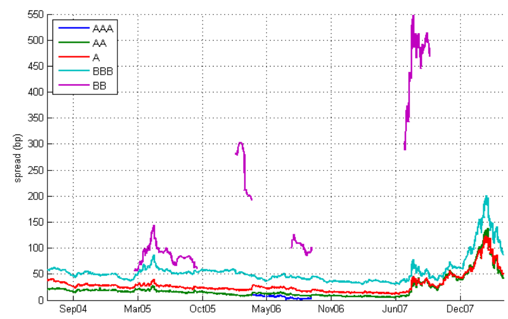
[1]On a computer with Intel Core Duo with 2.2GHz Processor.

three maturities is also plotted there. We used the data downloaded from the MorganMarkets[2], the internet-based data source of JP Morgan. These are the proprietary quotes of JP Morgan and not the official market quotes. The official quotes were not available for us. However, we don't find this a big problem since the bid-ask spreads on iTraxx are very small.

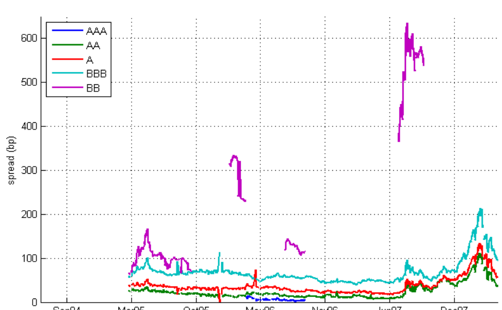
The data described so far is the basic iTraxx data necessary for any valuation model. This data would be even enough for the LHP model. For LHC model, a more specific rating-based data is needed. First of all, we need to know the rating composition of the iTraxx portfolio at any point in history. Unfortunately, such data is not available for download. We had to create the rating composition manually. All issuers ever been in the iTraxx Europe portfolio can be found in a convenient Excel format on the website of Markit[3]. We have created a rating history for all issuers and finally computed the rating composition for the iTraxx Europe portfolio on every trading day in the history. This rating history is presented in Figure 5. Most of the time the iTraxx Europe portfolio contained only issuers with ratings AA, A and BBB. The quote of the AA rating in the portfolio varied from 10 to 20%. Rating A was fluctuating around 40% and rating BBB around 45%. The rating AAA and BB were present in the portfolio only for short periods of time with a very small percentage.



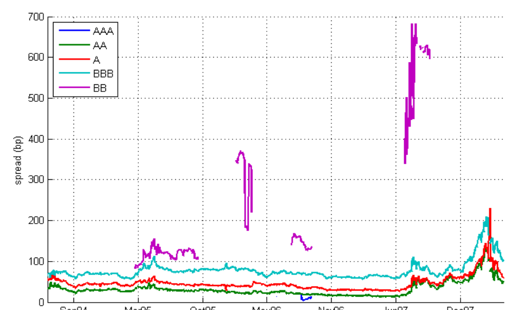
(a) Historical rating composition of the iTraxx portfolio



(b) Average 5-year spreads of the rating cells



(c) Average 7-year spreads of the rating cells



(d) Average 10-year spreads of the rating cells

Figure 5: Historical rating composition and average rating spreads for the iTraxx portfolio

[2]`https://mm.jpmorgan.com/redirect/bankone`

[3]Indices matrix available to download as an Excel sheet from
`http://www.markit.com/en/products/data/indices/credit-and-loan-indices/itraxx/matrix.page?`

The last data building block necessary for the LHC model are the average rating spreads. They are required to compute the rating specific default probabilities. As already noticed, the market average rating spreads are not appropriate for the LHC model since the rating cells of the iTraxx Europe portfolio are not large enough. Since the most tradable and liquid issuers are selected for the iTraxx Europe portfolio, they tend to be in the upper segment of the particular rating. So the average spreads of the iTraxx rating cells tend to be lower than the average European rating spreads.

As there exist no traded iTraxx Europe rating sub-indices, we had to compute them from the issuers' CDS data. In particular, we downloaded the data for the senior secured CDS with maturities of 5, 7 and 10 years for each issuer and constructed the average spreads for every rating at each point in time. The results are presented in Figure 5. Note that the average rating spreads are computed based on the issuers with the corresponding rating for the point in time. This is the reason why the AAA and BB spreads are available only partially.

The calibration of the Crash-NIG copula model can be performed in two steps. First, the Hidden Markov Model (HMM) is estimated separately. Since the correlation values are not observable, we have to use some other relevant observable process to estimate the model. This process follows two probability distributions depending on the state of the Markov chain. It is obvious from the plot of the historical iTraxx index spread data, that the states of the market have an impact not only on the correlations, but also on the iTraxx index spread. The spreads were very high during the observed crisis. We will use the 5 and 10 years iTraxx index spreads to derive the parameters of the Hidden Markov Model. We try to estimate the HMM with two distribution assumptions for the spread: normal and log-normal. If the Markov chain is in state one, the distribution of the spread or the log-spread is assumed to be $N(\mu_1, \sigma_1)$. In state two, the distribution is $N(\mu_2, \sigma_2)$. We also employed another process instead of the iTraxx spread, the base correlation of the equity tranche to calibrate the HMM. However, this data is only available since September 2004. We assume base correlation to be normally distributed. Note, that the base correlation is the implied correlation of the Gaussian copula model. The important difference between the iTraxx spread and the implied correlation of the equity tranche is that the iTraxx spread does not contain the correlation information directly. This information comes into the iTraxx spread indirectly, e.g., the correlation is typically high in the turbulent markets with high credit spreads. Since the implied correlation is a product of another credit portfolio model, we do not actually tend to use it in our model, but we employ this data series in the Hidden Markov Model estimation for the comparison and better understanding of the market.

- The transition matrix P is estimated with the help of the Baum-Welch Algorithm[4] which also provides the initial distribution and the parameters of the distribution of the observable process.

- Afterwards, we use the function Viterbi of the R package, that computes the most likely sequence of the states of the Markov chain given the estimated transition matrix, the initial distribution and the distribution parameters of the observable process.

- Using the transition matrix and the sequence of the most likely states, the probabilities $h_1^r(t)$ and $h_2^r(t)$ of the states one and two respectively on the time segments $[0, t]$, with $r \in 1, 2$ denoting the initial state at time 0, are computed according to Equation (25). These probabilities are calculated for all t, the time points of the premium payments.

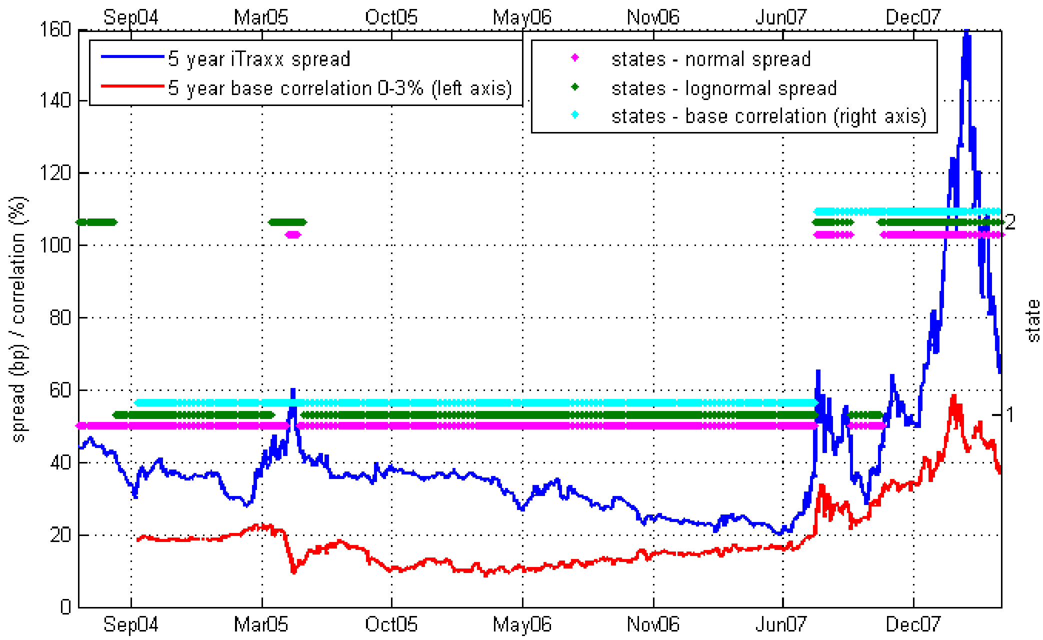
[4] We use the R package "HiddenMarkov"

In the second step, the probabilities $h_1^r(t)$ and $h_2^r(t)$ are used for the valuation of the iTraxx tranches and the optimization of the other model parameters. For this we use weekly data of the 5, 7 and 10 years iTraxx Europe tranches and compute the sum of the absolute error between the quoted and the model prices. For the tranches 3-6%, 6-9%, 9-12% and 12-22% the spreads are expressed in bp and the errors weighted with the weight 1. For the equity tranche 0-3%, the upfront fee is expressed in % and the error is weighted with the weight 0.1 to avoid its domination over the other tranches. For the optimization we use the Matlab function "fminsearch". Although, this is a local minimization algorithm, the convergence of the optimization problem is very good. We have tested it with different starting points and found the algorithm always converging to the same values.

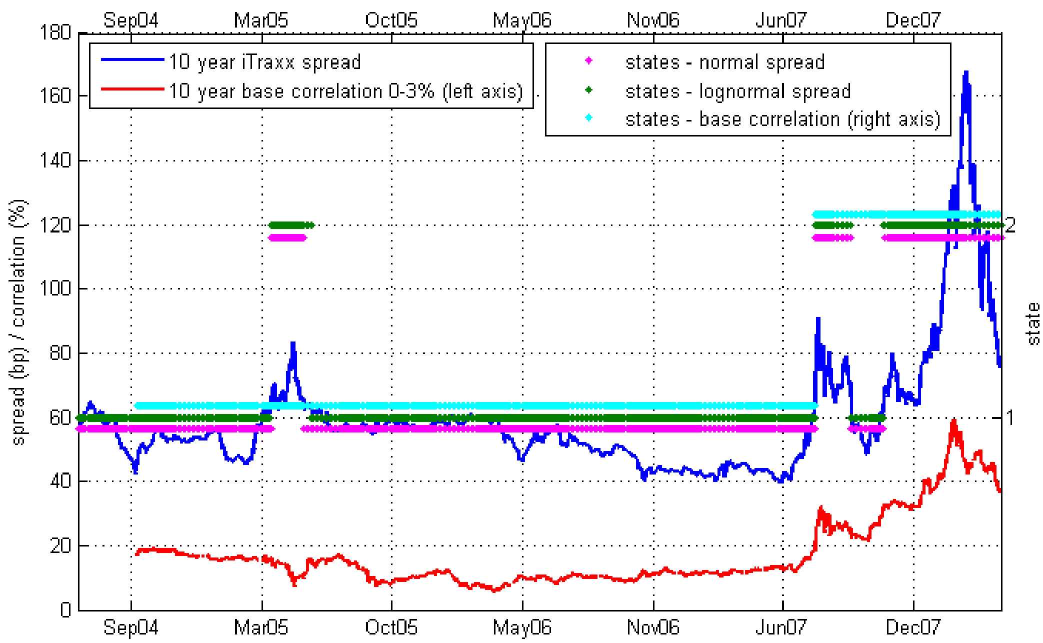
First of all, we take a look on Figure 6 presenting the Viterbi most likely sequence of the states based on the HMM model estimations with the 5 and 10 year spread and base correlation data. Besides of the three state sequences we have also plotted the iTraxx index spread and the base correlation of the equity tranche that were used to calibrate the Hidden Markov Model. Thus, we can better compare the change of the states with the evolution of the observed processes.

The state sequences based on the normal 5 and 10 year spreads and the log-normal 10-year spreads are similar. They all detect the second state during the market turbulences after the downgrade of Ford and General Motors in May, 2005, and during the sub-prime crisis starting in July, 2007, with a short break in September-October, 2007. The state sequences of the 5 and 10 year base correlations are identical and contain only a change from the first to the second state in July, 2007. This fact gives us a hint that the two crises are actually of a different nature and have different characteristics. We can find another confirmation of this in Figure 6 if we compare the evolutions of the iTraxx spread and the base correlation. The correlations were moving in the opposite direction than the spreads until July, 2007. Also, during the small crisis in May, 2005, the correlation was falling as the spread was growing. Since the beginning of the sub-prime crisis in July, 2007, the correlation changed its behavior. Since then it was growing or falling together with the spread. So, two states are actually not enough to describe the four year iTraxx history from June, 2004, to May, 2008. The two crisis that took place during this time were of different nature, the first one was a smaller branch-specific crisis while the second one has become a huge global market crisis. Hence, we try to calibrate the Crash-NIG copula model with three possible states.

The transition probabilities, initial distribution and the distribution parameters of the three state model estimated with the Baum-Welch algorithm are reported in the Table 2. In all cases the probabilities of staying in the same state are very high with more than 97%. The probability to stay in state three is equal to one in the case of the 5-year normal spread as well as 5 and 10 year correlation. All models except of the 5-year log-normal spread can change from state one only to state two directly.



(a) Calibration with the 5-year data



(b) Calibration with the 10-year data

Figure 6: Viterbi states in the two-states model

Table 2: HMM parameters of the three-state model

Data	Transition matrix	Initial distribution	Distribution parameters
5-year spread normal	$\begin{pmatrix} 0.9953 & 0.0047 & 0 \\ 0.0163 & 0.9787 & 0.005 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 31.25; & 5.54 \\ 46.50; & 5.92 \\ 99.34; & 26.69 \end{pmatrix}$
5-year spread lognormal	$\begin{pmatrix} 0.9863 & 0.0137 & 0 \\ 0.014 & 0.9773 & 0.0087 \\ 0.0125 & 0 & 0.9875 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3.28; & 0.14 \\ 3.60; & 0.05 \\ 4.10; & 0.37 \end{pmatrix}$
5-year 0-3% correlation normal	$\begin{pmatrix} 0.9925 & 0.005 & 0.0025 \\ 0.0069 & 0.9931 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0.17; & 0.01 \\ 0.11; & 0.02 \\ 0.36; & 0.09 \end{pmatrix}$
10-year spread normal	$\begin{pmatrix} 0.9843 & 0.0157 & 0 \\ 0.0110 & 0.9823 & 0.0067 \\ 0 & 0.0101 & 0.9899 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 45.61; & 3.22 \\ 56.54; & 2.99 \\ 85.72; & 25.56 \end{pmatrix}$
10-year spread lognormal	$\begin{pmatrix} 0.9843 & 0.0157 & 0 \\ 0.0112 & 0.9820 & 0.0068 \\ 0 & 0.0101 & 0.9899 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 3.82; & 0.07 \\ 4.03; & 0.05 \\ 4.41; & 0.26 \end{pmatrix}$
10-year 0-3% correlation normal	$\begin{pmatrix} 0.9986 & 0.0014 & 0 \\ 0 & 0.9908 & 0.0092 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0.12; & 0.03 \\ 0.28; & 0.04 \\ 0.45; & 0.06 \end{pmatrix}$

All models estimated with the spread data have the first state when the expected value of the spread is the lowest. The expected value of the spread is the highest in the third state. So the first state is the quiet state of the market, the second state is a bit turbulent and the third is the crisis state. This can also be seen in Figure 7 where the most likely states estimated with the Viterbi algorithm are plotted. However, only the states of the 5 years normal spread are exactly as discussed above: May 2005, is recognized as state two, together with June-August 2004 and July and September 2007. The rest of the history since December 2007 is estimated as the third state. The 5 and 10 year log-normal spreads and the 10-year normal spread put May 2005 to the same, third, state with the crisis after July 2007.

The states of the 5-year correlation results have exactly the characteristics we have discussed: the mean of the correlation is the lowest in the second state and the highest in the third state. Indeed, the Viterbi state in May 2005 of the 5-year correlation model is the second. However, different to the 5-year normal spread, this model has the second state also over a long time segment from September 2005 till October 2006. The third state starts already in July 2007. The 10-year correlation model states are different. Here, May 2005 is considered as a first, normal, state together with the complete time segment from August 2004 until July 2007. July to December 2007 is classified as the second state. And the spread explosion after December 2007 is recognized as the third state.

Before the calibration of the parameters of the other Crash-NIG copula model can be performed, the segment state probabilities must be computed. They are presented in Figure 8. These plots are organized in the following way. The first row gives the probabilities of the three states over the time segment $[0, t]$ with increasing t starting from the state one at time zero. The second row assumes that the initial state at time zero is two, and the third row the state three. The three Hidden Markov Models calibrated to the three data series are organized in the three columns: the calibration to the iTraxx spread with normal and log-normal distribution, and to the base correlation with normal distribution assumptions. For each time t on the x-axis of the plots, the black (grey) area gives the overall probability of the first (second) state during this time segment, and the white area, showing the probability of the third state, fills it up to one. The formulas for the probabilities are $\frac{h_i^r(t)}{t}$, with i denoting the initial state and r the state during the time period.

The effects of the transition matrices of the estimated Hidden Markov Models are clearer when looking at the segment state probabilities. First, we examine the probabilities of the 5 and 10 year log-normal spreads and the 10-year normal spread. Here, the segment probabilities of the three states, given different initial states, differ from one another only during the first 2 years. Afterwards, they converge very fast to asymptotic levels that are very similar for the three initial states. This means that the three states probably cannot have a big impact on the CDO pricing.

In contrast, the segment probabilities of the 5-year normal spread as well as of the both correlations are absolutely different, given different initial states. Starting in the first state, the probability of being in the first state during the time segment $[0, t]$ is decreasing continuously. The probability of the second state is almost the same, quite low, for any time segment, while the probability of the third state continuously increases. Given the initial state is the second, the picture is quite similar with much a higher probability of the second state at the beginning and the overall level of the first (third) state being lower (higher). For the 10-year correlation, the probability of the first state is even zero while the probability of the third state increases very fast up to almost one. Conditional on the third state as the initial one, the three models stay in this third state with probability one. Since the conditional state probabilities are so different for these models, we would expect the prices of the CDO tranches to be quite different from those of the model with only one state.

In a second step, we calibrated the parameters α, a_j, $j = 1, ..., 5$ and λ_1, λ_2 of the Crash-NIG copula model. The differences between the calibrated model and market spreads of all tranches were negative for numerous points in time. This means that the model spreads are much higher than the market quotes for all tranches. Actually, it is impossible to produce these low tranche spreads on such a day with any set of parameters and an individual calibration. The reason for this is far too high default probabilities implied by the rating average CDS spreads we have used.

On this point, it is time to note that so far we have not incorporated any liquidity premium into the model. Actually, the risk-neutral default probabilities were calculated from the average rating CDS spreads without any deduction for the liquidity premium. Valuation of the liquidity premium is a very complex subject with no good data sources. It is intuitive, that the liquidity premiums are much higher during a crisis. Liquidity premiums are also different for different markets: single CDS market, iTraxx index and iTraxx tranches. Our findings reflect that the liquidity premium in the single CDS market is higher than that of the iTraxx tranches. Liquidity premiums can be incorporated into the model in a simple way by deducing a fixed percentage of the credit spread. So we introduce the liquidity indicators l_r, with $r = 1, 2, 3$ the current state, such that the part of the credit spread representing the credit quality is l_r times the spread. Then the liquidity premium is $(1 - l_r)$ times the spread. Now the default probabilities that are used in the Crash-NIG copula model are not computed based on the complete spread but only on the part of it cleaned from the liquidity premium. We assume these liquidity indicators to be constant in the same state of the market and estimate them together with the other model parameters.

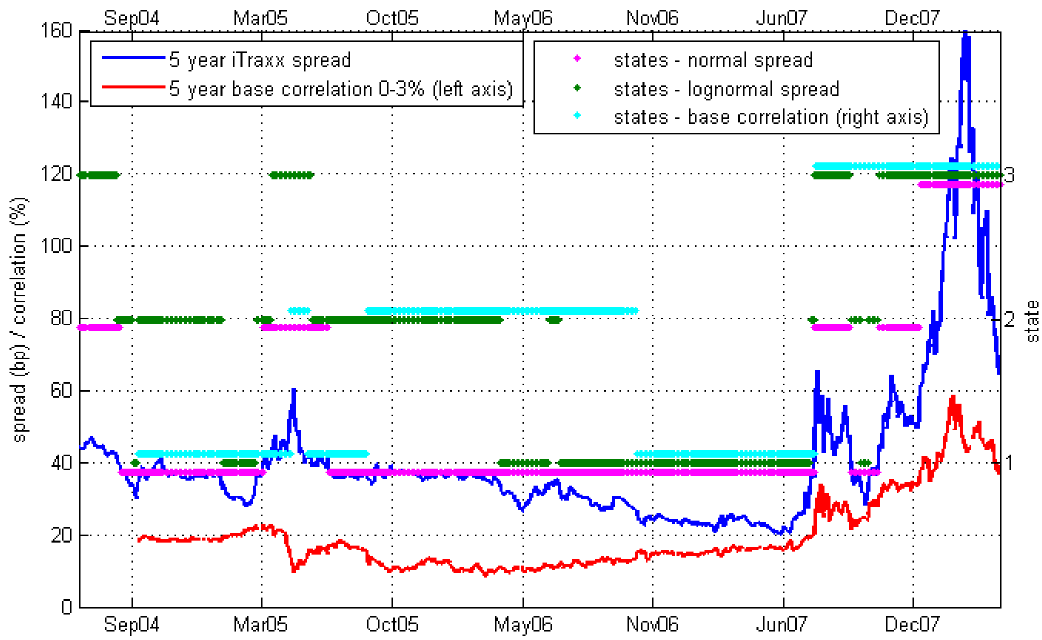
The estimated values of the parameters of the three state models are presented in Table 3. The parameter α of the NIG distribution is similar for all versions of the model. The five correlation parameters $a_1, ..., a_5$ do not differ much as well, having the highest value for a_2, the correlation parameter in the rating cell AA, and the lowest for a_5, in the rating cell BB. The two factors λ_1 and λ_2 giving the reduction or increase in correlation in the second and the third states vary across the models. The value of λ_1 is below one for all models meaning the reduction in correlation in the second state. The lowest value of 0.23 has the model with the 5-year normally distributed spread. The other three versions have much higher values of 0.63 to 0.73. The liquidity coefficients l_1, l_2, l_3 are similar for all the models. All the models, except of the 5-year log-normal spread, have the highest liquidity of 0.95-0.99 in the first state. The liquidity is a bit lower with 0.88-0.92 in the second state. The third state representing the global crash has the lowest liquidity of 0.73-0.77. In particular, this value means that around 23-27% of the credit spread is the liquidity premium and only 73-77% represent the price for the default protection.

Table 3: Parameters of the three-state Crash-NIG copula model

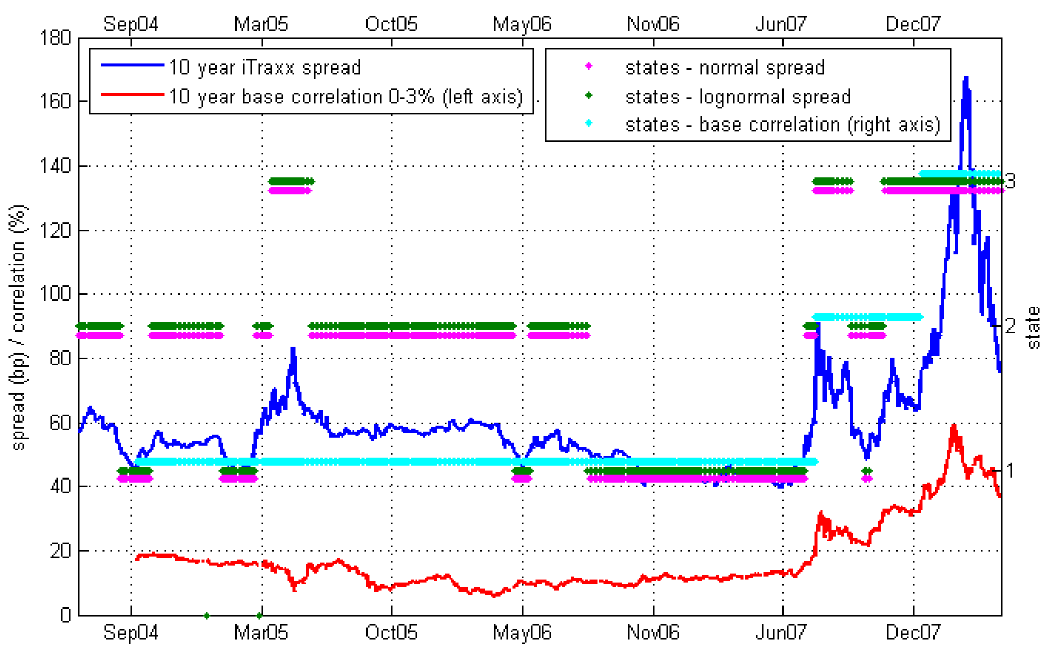
Data	5-year normal spread	5-year log-normal spread	10-year normal spread	10-year log-normal spread	one state model
α	0.3274	0.3287	0.3476	0.3728	0.3615
a_1	0.2562	0.3926	0.3167	0.4001	0.2476
a_2	0.5437	0.4247	0.4534	0.4168	0.9607
a_3	0.3429	0.4247	0.4461	0.4168	0.4975
a_4	0.2130	0.2150	0.2006	0.1899	0.3256
a_5	0.0828	0.1795	0.1739	0.1917	0.1161
λ_1	0.2353	0.6773	0.7375	0.6350	
λ_2	1.7443	2.2369	2.0920	2.2790	
l_1	0.9679	0.9122	0.9971	0.9792	0.9562
l_2	0.8827	0.9867	0.9394	0.9198	
l_3	0.7361	0.7679	0.7717	0.7589	
Aver. error (%)	14.80	18.99	19.08	19.05	23.98

Table 4: Average absolute deviations of tranche model spreads from the market spreads for the time period from 21.03.2005 to 08.05.2008: Crash-NIG copula model with 3 states and liquidity estimated with the 5-year normal spread

Maturity	0-3%	3-6%	6-9%	9-12%	12-22%
5 years	3.36%	19.62 bp	13.93 bp	10.52 bp	4.64 bp
7 years	4.43%	42.33 bp	19.34 bp	13.73 bp	7.62 bp
10 years	4.35%	61.36 bp	39.87 bp	23.21 bp	8.33 bp

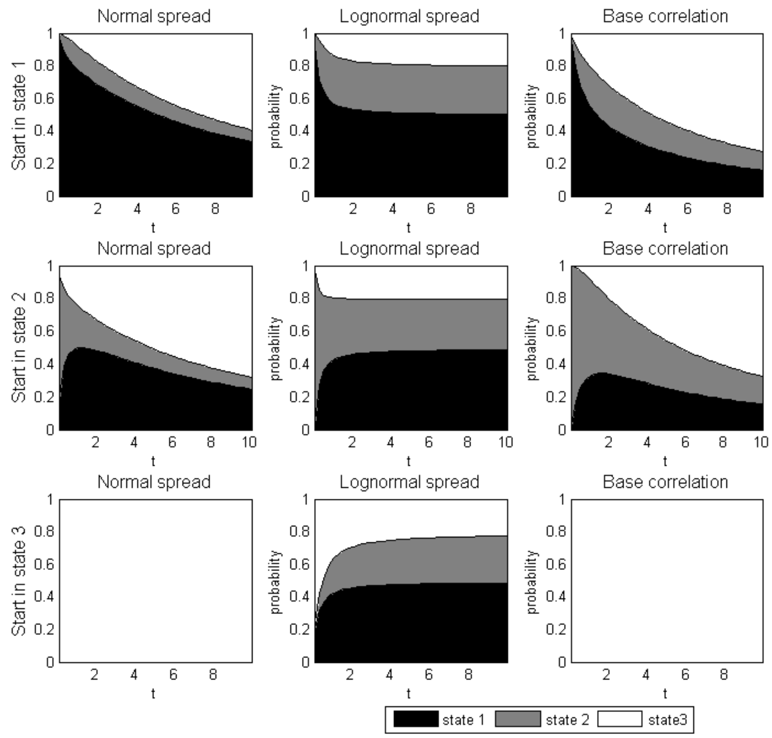


(a) Calibration with the 5-year data

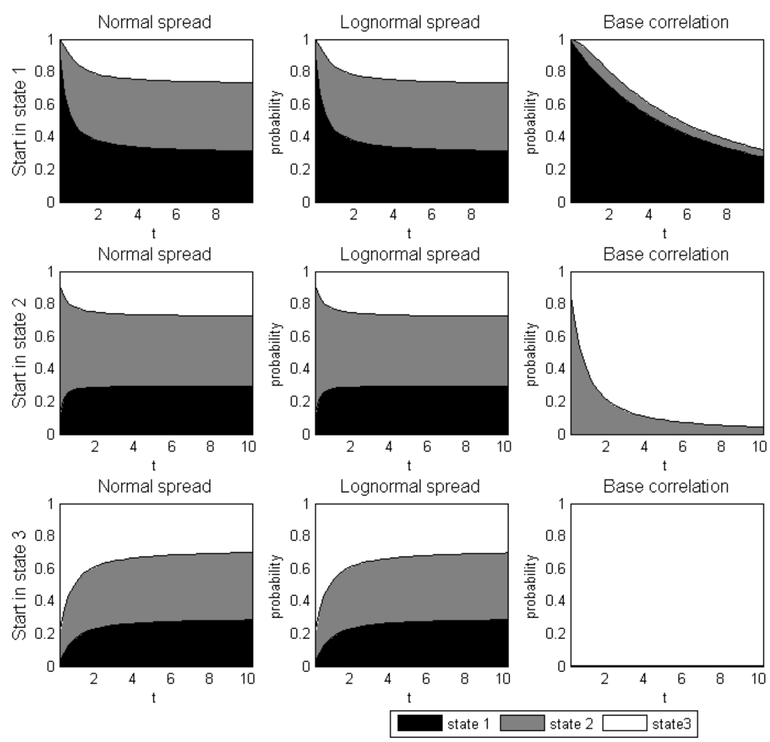


(b) Calibration with the 10-year data

Figure 7: Viterbi states in the three-states model



(a) Calibration with the 5-year data



(b) Calibration with the 10-year data

Figure 8: Probabilities of the states on increasing time segments

The reported errors are the average over all days in the history of sums of the absolute deviations of the model and market spreads over all tranches and maturities. The model estimated with the 5-year normally distributed spreads performs much better than the others. This is actually exactly what we expected, since the probabilities of being in different states, reported in Figure 8, for 5-year normal spread (similar to those estimated with the base correlation data) are very different for different initial states, while these probabilities for the other three models are only different for the first two years converging afterwards to the asymptotical values, that are very similar for all initial states. We have also calibrated the LHC NIG model with only one state to be able to evaluate the added value of the Crash-NIG copula model. The absolute error is much higher for this model than the error of all three-state models. The problem of the one-state model to deal with a crash scenario can be also seen on the values of the estimated correlation parameters. The correlation parameter a_2 of the AA rating cell is estimated to be 0.96, which is rather typical for the crash state and not for the normal correlation state. The average absolute deviations for each tranche and maturity are presented in Table 4 for the best performing model estimated with 5-year normal spreads. This is actually the breakdown of the overall error of 14.80% over the 15 tranches with different maturities. Figure 9 shows the quoted versus the model spreads. Here we can see that the fitting ability of the Crash-NIG copula model is very good. A model with more states could possibly fit the data even better, but one should also be aware of potential overfitting.

Conclusion

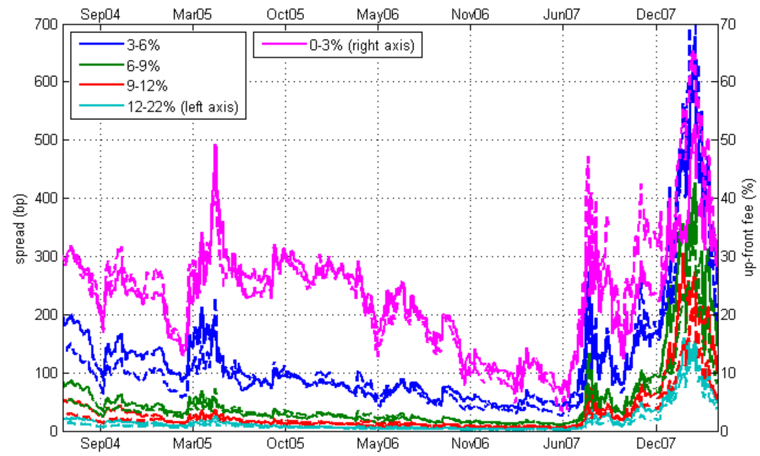
In this paper we have performed three enhancements of the NIG copula model presented by Kalemanova et al. (2007). Certainly, the one-factor copula models contain too many assumptions and thus cannot be used for an exact and detailed valuation of CDO structures, especially not for cash flow structures. However, this kind of models is very useful for other applications. In particular, these are risk management and measurement applications involving the generation of scenarios of the complete universe of market risk factors and the inclusion of CDO structures in a portfolio context. For this objective, it is necessary to have a simple and fast model that is also consistent with the scenario simulation framework.

The first enhancement introduces the time variable into the model and makes it consistent for the pricing of CDO tranches with different maturities. The new model is well defined for the purpose of scenario simulation since it can be discretized in an arbitrary way.

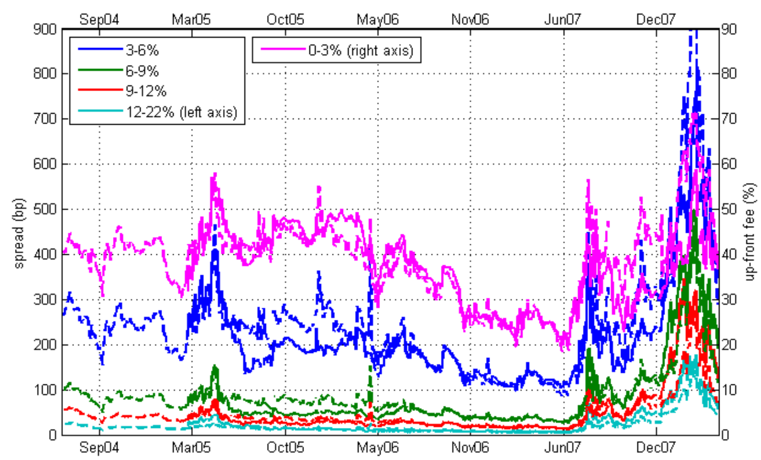
For the extension, we applied the LHC approach to the NIG model in order to bring more heteroscedasticity into the portfolio assumptions and to be able to use this model together with a simulation of rating migrations. The test calibration showed that the term-structure LHC NIG model could fit the data much better than the Gaussian LHC version.

Finally, we introduced the Crash-NIG copula model that allows for different correlation regimes. We demonstrated a calibration of the three-state version of the model to the history of iTraxx tranches, which showed that the fitting ability of the model is much better than that of a corresponding one-factor LHC NIG model.

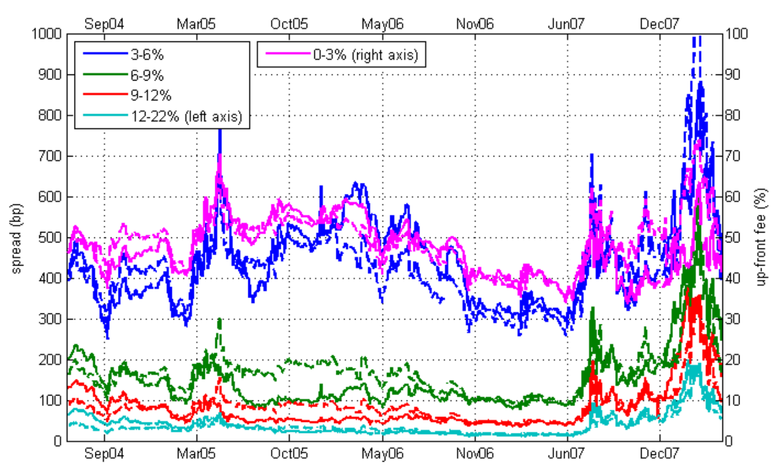
We also introduced liquidity premiums into the Crash-NIG copula model and showed that the actual credit crisis is substantially driven by liquidity effects.



(a) 5-year maturity



(b) 7-year maturity



(c) 10-year maturity

Figure 9: iTraxx quotes vs. three-state Crash-NIG copula model prices

A Proof of Proposition 3.1

Proof. We start with general NIG distribution for the factors:

$$\begin{aligned} d\widehat{M}(t) &\sim \mathcal{NIG}\left(\alpha^{CM},\, \beta^{CM},\, \mu^{CM},\, \delta^{CM}\right), && (28)\\ d\widehat{X_{ij}}(t) &\sim \mathcal{NIG}\left(\alpha^{CX},\, \beta^{CX},\, \mu^{CX},\, \delta^{CX}\right). && (29) \end{aligned}$$

It follows from the requirement (i) and the convolution property of the NIG distribution, that the first two parameters of the distributions must be equal to those in the first state:

$$\alpha^{CM} = \alpha,\; \beta^{CM} = \beta,\; \alpha^{CX} = \frac{\sqrt{1-a_j^2}}{a_j}\alpha,\; \beta^{CX} = \frac{\sqrt{1-a_j^2}}{a_j}\beta. \tag{30}$$

Besides, the requirement (iii) means that:

$$\mu^{CM} = -\delta^{CM}\frac{\beta}{\gamma},\; \mu^{CX} = -\delta^{CX}\frac{\beta}{\gamma}. \tag{31}$$

Now we consider the distribution of the asset return in the second state. This is the distribution of the sum of two NIG random variables:

$$\begin{aligned} \widehat{a}_j d\widehat{M}(t) &\sim \mathcal{NIG}\left(\frac{\alpha}{\widehat{a}_j},\, \frac{\beta}{\widehat{a}_j},\, -\widehat{a}_j\delta^{CM}\frac{\beta}{\gamma},\, \widehat{a}_j\delta^{CM}\right), && (32)\\ \sqrt{1-\widehat{a}_j^2}\,d\widehat{X_{ij}}(t) &\sim \mathcal{NIG}\left(\frac{\sqrt{1-a_j^2}}{a_j\sqrt{1-\widehat{a}_j^2}}\alpha,\, \frac{\sqrt{1-a_j^2}}{a_j\sqrt{1-\widehat{a}_j^2}}\beta,\right. && (33)\\ &\qquad \left. -\sqrt{1-\widehat{a}_j^2}\delta^{CX}\frac{\beta}{\gamma},\, \sqrt{1-\widehat{a}_j^2}\delta^{CX}\right). \end{aligned}$$

The both distributions are stable under convolution if the first two parameters are equal. This is the case when

$$\frac{1}{\widehat{a}_j} = \frac{\sqrt{1-a_j^2}}{a_j\sqrt{1-\widehat{a}_j^2}},$$

that is equivalent to

$$\frac{\sqrt{1-\widehat{a}_j^2}}{\widehat{a}_j} = \frac{\sqrt{1-a_j^2}}{a_j}.$$

This is only possible if

$$\widehat{a}_j = a_j. \tag{34}$$

So the distribution of the asset return in the second state is given by

$$dA_{ij}(t) \sim \mathcal{NIG}\left(\frac{\alpha}{a_j},\, \frac{\beta}{a_j},\, -\left(a_j\delta^{CM} + \sqrt{1-a_j^2}\delta^{CX}\right)\frac{\beta}{\gamma},\, a_j\delta^{CM} + \sqrt{1-a_j^2}\delta^{CX}\right). \tag{35}$$

According to the requirement (ii), the distribution must be the same as in the first state, i.e. the third and the fourth parameter must be:

$$\begin{aligned} -\left(a_j\delta^{CM} + \sqrt{1-a_j^2}\delta^{CX}\right)\frac{\beta}{\gamma} &= -\frac{1}{a_j}\frac{\beta\gamma^2}{\alpha^2}dt, && (36)\\ a_j\delta^{CM} + \sqrt{1-a_j^2}\delta^{CX} &= \frac{1}{a_j}\frac{\gamma^3}{\alpha^2}dt && (37) \end{aligned}$$

The two equations are actually the same. So we have only one equation to solve for two variables δ^{CM} and δ^{CX}. We also have the last requirement (iv) that still is not satisfied. So we look at the parameter δ^{CM}:

$$\delta^{CM} = \frac{1}{a_j^2}\frac{\gamma^3}{\alpha^2}dt - \frac{\sqrt{1-a_j^2}}{a_j}\delta^{CX}.$$

δ^{CM} can be independent of a_j only if it has the form

$$\delta^{CM} = k\frac{\gamma^3}{\alpha^2}dt,$$

for some constant $k > 0$. Then corresponding δ^{CX} is

$$\delta^{CX} = \frac{1-ka_j^2}{1-a_j^2}\frac{\sqrt{1-a_j^2}}{a_j}\frac{\gamma^3}{\alpha^2}dt,$$

and we come up with the distributions in Equation (16) that completes the proof. □

B Proof of Proposition 3.3

Proof. For the i-th central moment of the uncondition distribution of $M(t)$ we have:

$$\begin{aligned}
&\int_{-\infty}^{\infty} x^i f_{M(t)}(x)dx \\
= &\int_{-\infty}^{\infty}\int_{\Omega_t} x^i f_{\mathcal{NIG}}\left(x;\alpha,\,\beta,\,-\left(z_1+\lambda^2 z_2\right)\frac{\beta\gamma^2}{\alpha^2},\,\left(z_1+\lambda^2 z_2\right)\frac{\gamma^3}{\alpha^2}\right) f_{T^r(t)}(z_1,z_2)d(z_1,z_2)dx \\
= &\int_{\Omega_t}\left[\int_{-\infty}^{\infty} x^i f_{\mathcal{NIG}}\left(x;\alpha,\,\beta,\,-\left(z_1+\lambda^2 z_2\right)\frac{\beta\gamma^2}{\alpha^2},\,\left(z_1+\lambda^2 z_2\right)\frac{\gamma^3}{\alpha^2}\right)dx\right] f_{T^r(t)}(z_1,z_2)d(z_1,z_2).
\end{aligned}$$

So we must just integrate the moments of $M(t)|T^r(t)$, i.e. the moments of a NIG distribution, over the distribution of the duration stays. Since the expectation of the conditional distribution of $M(t)|T^r(t)$ is zero, the unconditional expectation is zero as well, i.e. $\mathbb{E}\left(M(t)\right) = 0$. Further, the variance of $M(t)$ is:

$$\mathbb{V}\left(M(t)\right) = \mathbb{E}\left(\left(T_1^r(t)+\lambda^2 T_2^r(t)\right)\frac{\gamma^3}{\alpha^2}\frac{\alpha^2}{\gamma^3}\right) = \mathbb{E}\left(T_1^r(t)+\lambda^2 T_2^r(t)\right).$$

The skewness of $M(t)$ is:

$$\mathbb{S}\left(M(t)\right) = \mathbb{E}\left(\frac{3\beta}{\alpha\sqrt{\gamma}\sqrt{\left(T_1^r(t)+\lambda^2 T_2^r(t)\right)\frac{\gamma^3}{\alpha^2}}}\right) = \frac{3\beta}{\gamma^2}\mathbb{E}\left(\frac{1}{\sqrt{T_1^r(t)+\lambda^2 T_2^r(t)}}\right).$$

And finally, the kurtosis of $M(t)$ is:

$$\begin{aligned}
\mathbb{K}\left(M(t)\right) &= 3+3\left(1+4\left(\frac{\beta}{\alpha}\right)^2\right)\mathbb{E}\left(\frac{1}{\gamma\left(T_1^r(t)+\lambda^2 T_2^r(t)\right)\frac{\gamma^3}{\alpha^2}}\right) \\
&= 3+3\left(1+4\left(\frac{\beta}{\alpha}\right)^2\right)\mathbb{E}\left(\frac{1}{T_1^r(t)+\lambda^2 T_2^r(t)}\right)\frac{\alpha^2}{\gamma^4}.
\end{aligned}$$

The derivation of the moments for $X_{ij}(t)$ is analogue and straightforward. $\square$

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