Analytic Models of the ROC Curve: Applications to Credit Rating Model Validation

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Abstract: In this paper, the authors use the concept of the population Receiver Operating Characteristic (ROC) curve to build analytic models of ROC curves. Information about the population properties can be used to gain greater accuracy of estimation relative to the non-parametric methods currently in vogue. If used properly, this is particularly helpful in some situations where the number of sick loans is rather small, a situation frequently met in practice and in periods of benign macro-economic background.

Keywords: Validation, Credit Analysis, Rating Models, ROC, Basel II

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1 Introduction

Following the publication in June 2004 of the ‘International Convergence of Capital Measurement and Capital Standards: A Revised Framework’ by the Basel Committee on Banking Supervision (BCBS) (known as Basel II), qualified banks are now allowed to use the Internal Rating-Based (IRB) credit risk modelling approach for risk modelling and economical capital calculation. One of the important components of most IRB risk models is the rating system used for transforming and assigning the Probability of Default (PD) to each obligor in the credit portfolio, and over the last three decades, banks and public ratings agencies have developed a variety of rating methodologies. Therefore, questions arise as to which of these methodologies deliver acceptable discriminatory power between the defaulting and non-defaulting obligor *ex ante*, and which methodologies would be preferred for different obligor sub-groups. It has become increasingly important for both the regulator and the banks to quantify and judge the quality of rating systems.

This concern is reflected and stressed in the recent BCBS working paper No.14 (2005) which summarizes a number of statistical methodologies for assessing discriminatory power described in the literature. For example, Cumulative Accuracy Profile (CAP), Receiver Operating Characteristic (ROC), Bayesian error rate, Conditional Information Entropy Ratio (CIER), Kendall’s τ and Somers’ D, Brier score, *inter alia*. Among those methodologies, the most popular ones are CAP and its summary index, the Accuracy Ratio (AR), as well as ROC and its summary index known as the Area under the ROC (AUROC). It is worth noting that, unlike some other measures that do not take sample size into account and are therefore substantially affected by statistical errors, the CAP and the ROC measures explicitly account for the size of the default sample and, thus, can be used for direct rating model comparison.

A detailed explanation of the CAP is presented in Sobehart, Keenan and Stein (2000), and Sobehart and Keenan (2004). ROC has long been used in medicine, psychology and signal detection theory, so there is a large body of literature that analyses the properties of the ROC curve. Bamber (1975) shows that the AUROC is related to the Mann-Whitney Statistic, and also discusses several different methods for constructing confidence intervals. An overview of possible applications of the ROC curves is given by Swets (1988). Sobehart and Keenan (2001) introduce the ROC concept to internal rating model validation and focus on the calculation and interpretation of the ROC measure. Engelmann, Hayden and Tasche (2003) show that AR is a linear transformation of AUROC; their work complements the work of Sobehart and Keenan (2001) with more statistical analysis of the ROC. However, the previous work with which we are familiar has used a finite sample of empirical or simulated data, but no-one has analyzed the analytic properties of the ROC Curve and the ROC measure under parametric assumptions for the distribution of the rating scores.

In this paper, we further explore the statistical properties of the ROC Curve and its summary indices, especially under a number of rating score distribution assumptions. We focus on the analytical properties of the ROC Curve alone since the CAP measure is just a linear transformation of the ROC measure.

In section 2, in order to keep this paper self-contained, we briefly introduce the credit rating model validation background and explain the concepts and definitions of ROC and CAP. A general equation for the ROC Curve is derived. By assuming the existence of probability density functions of the two variables that construct the ROC Curve, an unrestrictive assumption, we show that there is a link between the first derivative of the curve and the likelihood ratio of the two variables, a result derived by different methods in Bamber (1975).

In section 3, by further assuming certain statistical distributions for the credit rating scores, we derive analytic solutions for the ROC Curve and its summary indices respectively. In particular, when the underlying distributions are both Negative Exponential distributions, we have a closed form solution.

In section 4, we apply the results derived in section 3 to simulated data. Performance evaluation reports are presented. Section 5 concludes:

We find that estimation results from our analytic approach are as good as and, in many cases, better than the non-parametric AUROC ones. The accuracy of our approach is limited by the continuous rating score assumption, and also affected by the accuracy of estimation of the distribution parameters on rating score samples in some cases. However, it offers direct insight into more complex situations and is, we argue, a better tool in credit rating model selection procedure, since the analytic solution can be used as an objective function.

2 Theoretical Implication and Applications:

In this section, we first briefly review the credit rating system methodology, in particular, the CAP and the ROC measures. The content presented in this part is very similar to Engelmann, Hayden and Tasche (2003) and BCBS working paper No.14 (2005). Then we introduce the Ordinary Dominance Graph (ODG) where ROC is a special case of ODG, and some interesting theoretical implications of the ROC curve will be given.

2.1 The validation of credit rating system.

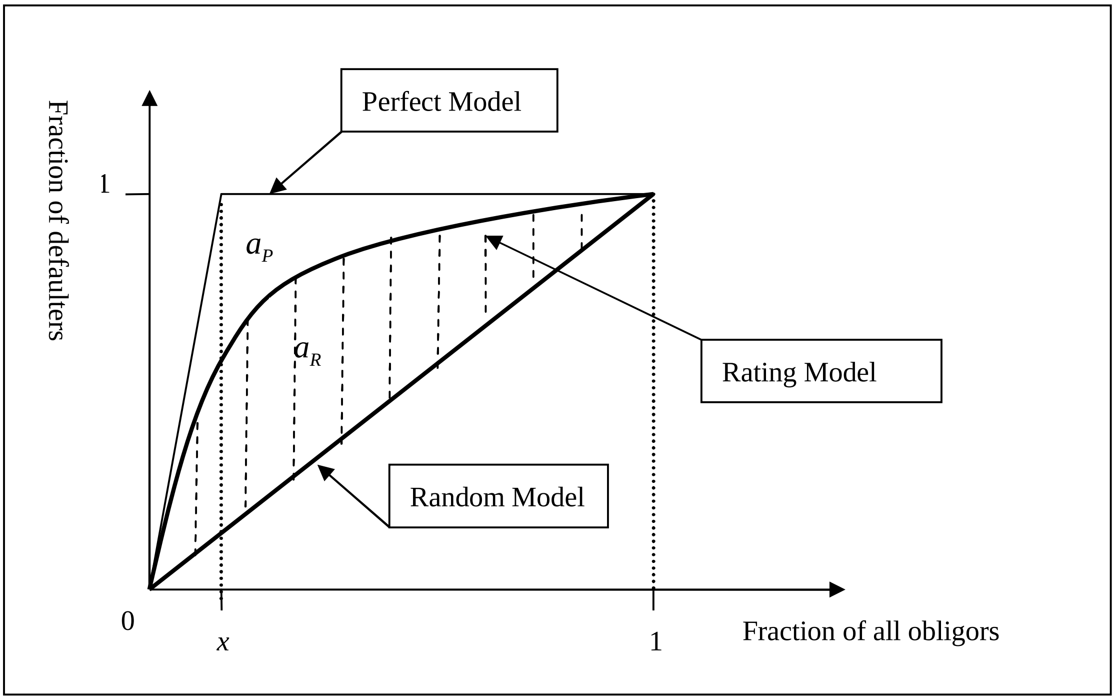
The statistical analysis of rating models is based on the assumption that for a predefined time horizon, there are two groups of bank obligors: those that will be in default, called defaulters, and those that will not be in default, called non-defaulters. It is not observable in advance whether an obligor will be a defaulter or a non-defaulter in the next time horizon. Banks have loan books or credit portfolios; they have to assess an obligor's future status based on a set of his or her present observable characteristics. Rating systems may be regarded as classification tools to provide signals and indications of the obligor's possible future status. A rating score is returned for each obligor based on a rating model, usually an Expert Judgment Model. The main principal of rating systems is that “the better a grade, the smaller the proportion of defaulters and the greater the proportion of non-defaulters that are assigned this grade”. Some quantitative examples are the famous Altman’s Z score, or some scores from a Logit model.

Therefore, the quality of a rating system is determined by its discriminatory power between non-defaulting obligors and defaulters *ex ante* for a specific time horizon, usually a year. The CAP measure and ROC provide statistical measures to assess the discriminatory power of various rating models based on historical (*ex post*) data.

2.2 Cumulative Accuracy Profile and Accuracy Ratio

Consider an arbitrary rating model that produces a rating score, where a high score is usually an indicator of a low default probability. To obtain the CAP curve, all debtors are first ordered by their respective scores, from riskiest to safest, i.e. from the debtor with the lowest score to the debtor with the highest score. For a given fraction x of the total number of debtors the CAP curve is constructed by calculating the percentage $d(x)$ of the defaulters whose rating scores are equal to, or lower than, the maximum score of fraction x. This is done for x ranging from 0% to 100%. Figure 1 illustrates CAP curves.

Figure 1. Cumulative Accuracy Profile



A perfect rating model will assign the lowest scores to the defaulters. In this case the CAP increases linearly to one, then remains constant. For a random model without any discriminative power, the fraction *x* of all debtors with the lowest rating scores will contain *x* percent of all defaulters. The real rating system lies somewhere in between these two extremes. The quality of a rating system is measured by the Accuracy Ratio (AR). It is defined as the ratio of the area a_R between the CAP of the rating model being validated and the CAP of the random model, and the area a_P between the CAP of the perfect rating model and the CAP of the random model.

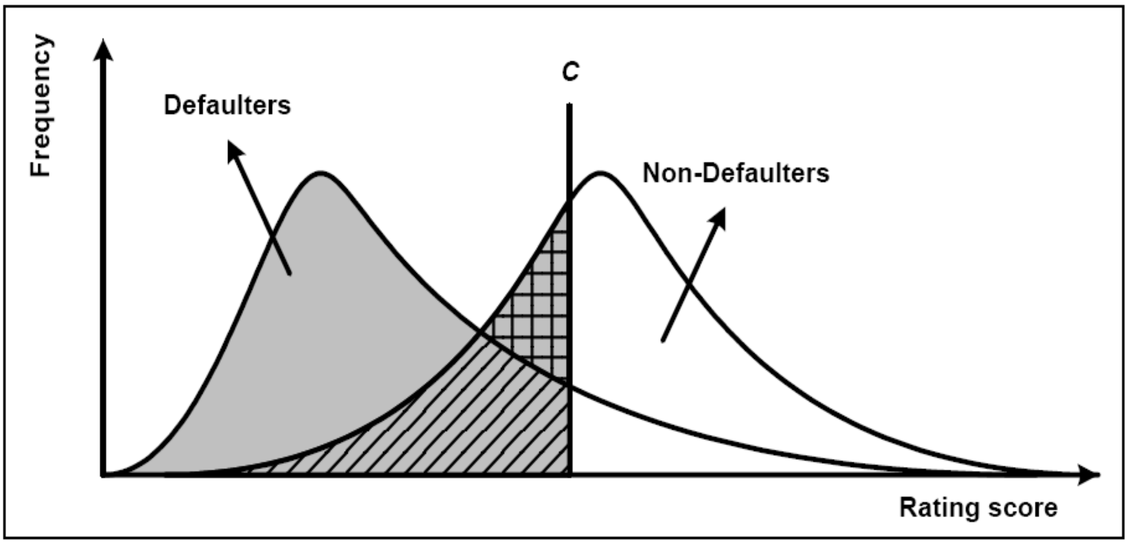
$$AR = \frac{a_R}{a_P}$$

It is easy to see that for real rating models the AR ranges from zero to one, and the closer the AR is to one, the better the rating model,.

2.3 Receiver Operating Characteristic and the Area under the ROC curve

The construction of a ROC curve is illustrated in Figure 2 which shows possible distributions of rating scores for defaulting and non-defaulting debtors. For a perfect rating model the left distribution and the right distribution in Figure 2 would be separate. For real rating systems, perfect discrimination in general is not possible. Distributions will overlap as illustrated in Figure 2. (reproduced from BCBS Working paper No.14, (2005))

Figure 2: Distribution of rating scores for defaulting and non-defaulting debtors



Assume one has to use the rating scores to decide which debtors will survive during the next period and which debtors will default. One possibility for the decision-maker would be to introduce a cut-off value *C* as in Figure 2, then each debtor with a rating score lower than *C* is classed as a potential defaulter, and each debtor with a rating score higher than *C* is classed as a non-defaulter. Four decision results would be possible. If the rating score is below the cut-off value *C* and the debtor subsequently defaults, the decision was correct. Otherwise the decision-maker wrongly classified a non-defaulter as a defaulter. If the rating score is above the cut-off value and the debtor does not default, the classification was correct. Otherwise a defaulter was incorrectly assigned to the non-defaulters' group.

Then one can define a hit rate *HR(C)* as:

$$HR(C) = \frac{H(C)}{N_D}$$

where *H(C)* is the number of defaulters predicted correctly with the cut-off value *C*, and N_D is the total number of defaulters in the sample. This means that the hit rate is the fraction of defaulters that were classified correctly for a given cut-off value *C*. The false alarm rate *FAR(C)* is then defined as:

$$FAR(C) = \frac{F(C)}{N_{ND}}$$

where *F(C)* is the number of false alarms, i.e. the number of non-defaulters that were classified incorrectly as defaulters by using the cut-off value *C*. The total number of non-defaulters in the sample is denoted by N_{ND} . In Figure 2, *HR(C)* is the area to the left of the cut-off value *C* under the score distribution of the defaulters (coloured plus hatched area), while *FAR(C)* is the area to the left of *C* under the score distribution of the non-defaulters (chequered plus hatched area).

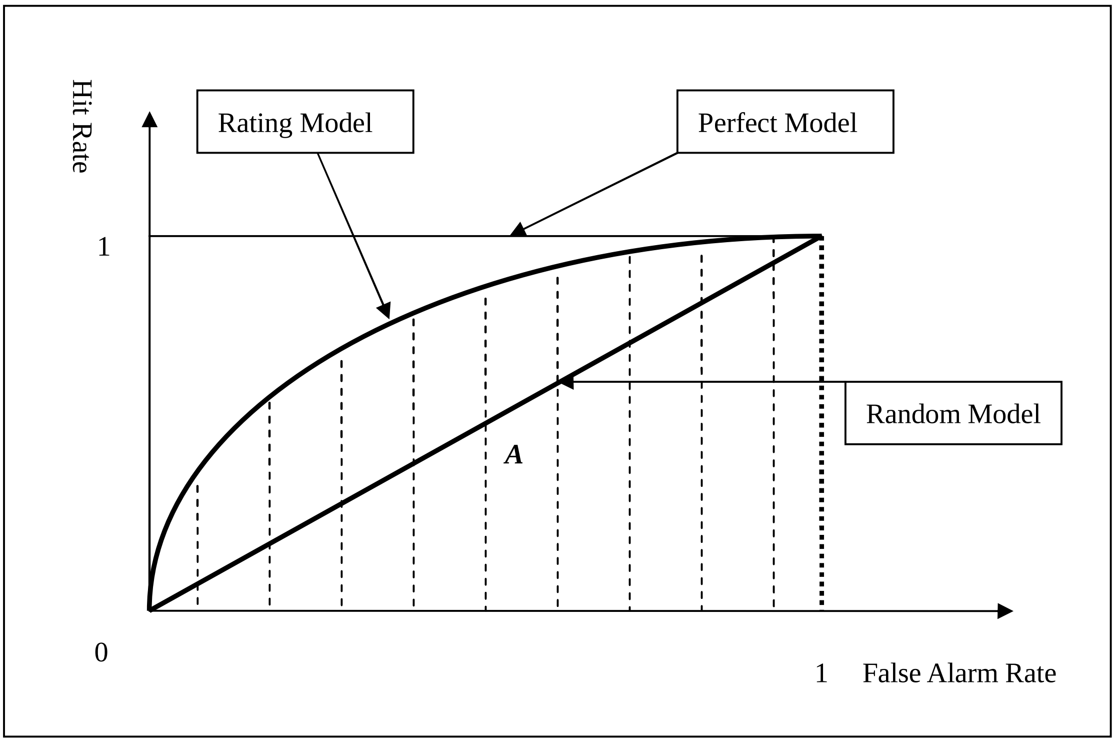
To construct the ROC curve, the quantities *HR(C)* and *FAR(C)* are computed for all possible cut-off values of *C* that are contained in the range of the rating scores; the ROC curve is a plot of *HR(C)* versus *FAR(C)*, illustrated in Figure 3.

The accuracy of a rating model's performance increases the steeper the ROC curve is at the left end, and the closer the ROC curve's position is to the point (0,1). Similarly, the larger the area under the ROC curve, the better the model. This area is called AUROC and is denoted by *A*. By means of a change of variable, it can be calculated as

$$A = \int_0^1 HR(FAR)d(FAR)$$

The area *A* is 0.5 for a random model without discriminative power and it is 1.0 for a perfect model. In practice, it is between 0.5 and 1.0 for any reasonable rating model.

Figure 3: Receiver Operating Characteristic Curves.



It has been shown in Engelmann, Hayden and Tasche (2003) that:

$$AR = \frac{a_R}{a_P} = \frac{N_{ND}\left(A-0.5\right)}{0.5N_{ND}} = 2\left(A-0.5\right) = 2A-1$$

2.4 Some further statistical properties of ROC measures

The ROC stems from the Ordinal Dominance Graph (ODG). Assume we have two sets of continuous random variables, *X* and *Y*. Let C be an arbitrary constant. Define:

$$y = prob\left(Y \le C\right) = F_Y\left(C\right) \text{ and } x = prob\left(X \le C\right) = F_X\left(C\right)$$

where *x* and *y* lie between [0, 1] and C lies in $\left(-\infty, +\infty\right)$. Then the ODG is simply a plot of *y* against *x*. See Figure 4. There are some straight forward properties of the ODG:

1) The ODG curve is never decreasing, as *x* increases *y* cannot decrease.
2) If $Prob(Y \le C) = Prob(X \le C)$, then *x* and *y* are identically distributed, *y* = *x* and the ODG curve is a $45°$ line.

3) If X first order stochastic dominates (FSD) Y, then the ODG curve lies above the $45°$ line and vice versa.

Proof:

If X FSD $Y \Rightarrow F_X(C) \le F_Y(C), \quad \forall C \in \mathbb{R} \quad \Rightarrow x \le y$

$\Rightarrow (x, y)$ lies above the 45° line

If we regard y as score signals of defaulters in the next predefined period and x as those of the non-defaulters, then we expect any sensible rating system to produce *Prob*($Y \le C$) $\geq$ *Prob*($X \le C$) for all C. Thus $x \le y$ for all C and the ODG curve is above the $45°$ line. In the risk literature, it is referred to as the ROC curve. In terms of section 2.3, y is the *HR(C)* and x is the *FAR(C)*.

By assuming the existence of probability density functions (PDF) of F_X and F_Y, i.e. that they are both absolutely continuous, the following lemma can be derived:

Lemma 1:

If $x = F_X(C), C = F_X^{-1}(x)$, then $\frac{\partial C}{\partial x} = \frac{1}{f_X(C)}$, where $f_X(C)$ is the PDF of X..

Proof: $1 = \frac{\partial F_X(C)}{\partial x} = \frac{\partial F_X(C)}{\partial C} \cdot \frac{\partial C}{\partial x} = f_X(C)\frac{\partial C}{\partial x}, \ \therefore \frac{\partial C}{\partial x} = \frac{1}{f_X(C)}$

Lemma 2:

If $y = F_Y(C)$, then $\frac{\partial y}{\partial x} = \frac{f_Y(C)}{f_X(C)}$.

We see from Lemma 2 that the slope of the ODG curve is just the likelihood ratio (LR) of Y and X evaluated at C.

Proof: $\frac{\partial y}{\partial x} = \frac{\partial F_Y(C)}{\partial x} = \frac{\partial F_Y(C)}{\partial C} \cdot \frac{\partial C}{\partial x} = f_Y(C) \cdot \frac{\partial C}{\partial x} = \frac{f_Y(C)}{f_X(C)}$

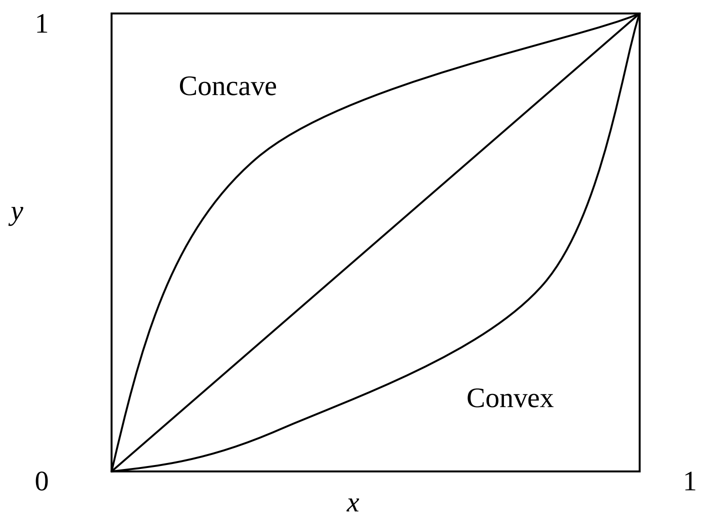
Theorem:

If $\dfrac{f_Y(C)}{f_X(C)}$ is increasing in C, then $\dfrac{\partial y}{\partial x}$ is increasing and the ODG curve is convex.

If $\dfrac{f_Y(C)}{f_X(C)}$ is decreasing in C, then $\dfrac{\partial y}{\partial x}$ is decreasing and the ODG curve is concave.

The latter case is the one that we are interested in, as it is the ROC curve. See Figure 4.

Figure 4: Ordinal Dominance Graph



We have assumed that *X* and *Y* have PDF's, thus they are continuous random variables. Then, AUROC can then be expressed as:

$$A = \text{Prob}(y \le x) = \int_{-\infty}^{\infty} \text{Prob}\left(Y \le X \mid X = C\right)\text{Prob}\left(X = C\right) dC$$

Since *X* and *Y* are scores from different obligor groups, they are independent.

We have: $\text{Prob}\left(Y \le X \mid X = C\right) = \text{Prob}\left(Y \le C\right)$.

Since $y = F_Y(C) = \text{Prob}\left(Y \le C\right)$ and $\partial x = f_X(C)\partial C$ $\Longrightarrow$

$$A = \int_{-\infty}^{\infty} F_Y(\text{C})\, f_X(C)\, dC = \int_0^1 F_Y\left(F_X^{-1}(x)\right) dx \quad \text{------------------------------ (1)}$$

A modelling exercise may choose a rating model that maximizes the AUROC with respect to the obligor group under review. But how would one estimate Prob($y \leq x$)? Bamber(1975) and Engelmann, Hayden and Tasche (2003) show that given a rating-scores sample of the obligors assigned by a rating model, the AUROC could be estimated in an non-parametric way using the Mann-Whitney U statistic.

On the other hand, if we had a parametric distribution model for X and Y, then we could explicitly derive a formula for the ROC curve and for the AUROC. In the next section, we will review some plausible distributions for X and Y and derive the closed form solutions wherever possible.

3 Choices of Distributions

In this section, we derive analytical formulae for the ROC by assuming that the rating scores produced by rating models follow plausible distributions. The distributions we present here are Weibull Distribution (including Exponential Distribution), Logistic Distribution, Normal Distribution and Mixed models for X and Y respectively. Where we have explicit closed forms for the ROC curve, we derive the closed form AUROC as well. The case of mixed distributions for X and Y can easily be extended from the following discussion.

We use the symbol M for the location parameters (sometimes the mean, sometimes the minimum), λ for the scale parameter and α for the shape.

3.1 Weibull Distribution

We first present solutions under a Weibull Distribution assumption of rating scores. The Weibull Distribution is flexible and rich. A three-parameter Weibull distribution cumulative probability function (CDF) is given by:

$$F(z) \equiv P(Z \le z) = 1 - e^{-\left(\frac{z-M}{\lambda}\right)^{\alpha}},$$

where $z>M$, $\alpha>0$ and $\lambda>0$ The inverse CDF of a three parameter Weibull Distribution is:

$$F^{-1}(p) = M + \lambda\left[-\ln(1-p)\right]^{\frac{1}{\alpha}},$$

where $p \in [0,1]$. Assuming y is the *HR(C)* and x is the *FAR(C)*, the three-parameter Weibull Distribution ROC is derived as:

$$x = F_X(C) = 1 - e^{-\left(\frac{C-M_x}{\lambda_x}\right)^{\alpha_x}} \Rightarrow C = M_x + \lambda_x\left[-\ln(1-x)\right]^{\frac{1}{\alpha_x}}$$

$$y = F_Y(C) = P(Y \le C) = 1 - e^{-\left(\frac{C-M_y}{\lambda_y}\right)^{\alpha_y}}$$

$$\text{ROC}: y = 1 - \exp\left[-\left(\frac{\lambda_x}{\lambda_y}\left[-\ln(1-x)\right]^{\frac{1}{\alpha_x}} + \left(M_x - M_y\right)\right)^{\alpha_y}\right] \text{ --------------------------- (2)}$$

The above ROC formula is very difficult to use for deducing an explicit closed form formula for the AUROC, although a numerical solution exists in this case once all the parameters are estimated. However, the situation becomes much better if we impose the slightly stronger assumption that the shape parameters α of the F_X and F_Y are equal to one. We then have analytical closed form solutions and the Weibull distribution degenerates to a truncated Exponential Distribution Family if M is positive and an extended Exponential if M is negative.

Assume: $\alpha_x = \alpha_y = 1$, we then rewrite the above equation as following:

$$\text{ROC}: y = 1 - e^{\frac{M_y - M_x}{\lambda_y}}(1-x)^{\frac{\lambda_x}{\lambda_y}}$$

Let $K = e^{\frac{M_y - M_x}{\lambda_y}}$, $\theta = \frac{\lambda_x}{\lambda_y}$, then

$$AUROC = \int_0^1\left[1 - K(1-x)^{\theta}\right]dx = 1 - \frac{K}{1+\theta} = 1 - \frac{\lambda_y}{\lambda_x + \lambda_y}e^{\frac{M_y - M_x}{\lambda_y}}, \text{ ----------------------- (3)}$$

We next discuss some of the properties of equation 3.

Property 1. for the 2 parameter Weibull model:

$M_y - M_x \in (-\infty, 0)$ for any plausible rating system. The smaller $M_y - M_x$ is (or the larger $|M_y - M_x|$ is), the closer the AUROC is to one. Recall that M is the location parameter, in this case the minimum. Therefore, the rating system will receive a higher AUROC if it can better discriminate the defaulter from non-defaulter by the difference in their minimum values.

It is also interesting to see that if $M_y - M_x \to 0$, $\text{AUROC} \to 1 - \frac{\lambda_y}{\lambda_x + \lambda_y}$. As we illustrated in an earlier section, AUROC of a plausible, non-random rating system is above 0.5 . This implies that the value of the scale parameters to which the rating scores are assigned have to be such that $0 < \lambda_y \le \lambda_x$. Note that this condition is implied where both groups are exponential but also where both groups are truncated or extended exponentials with the same minima.

Property 2. for the 2 parameter Weibull model:

AUROC is monotonically increasing with respect to λ_x, but monotonically decreasing with respect to λ_y.

Proof:

$$\frac{d}{d\lambda_y} AUROC = \frac{K}{(\lambda_x + \lambda_y)^2 \lambda_y} \left[(M_y - M_x)(\lambda_x + \lambda_y) - \lambda_y (\lambda_x + 2\lambda_y) \right]$$

$$\left.\begin{array}{l} \left[(M_y - M_x)(\lambda_x + \lambda_y) - \lambda_y (\lambda_x + 2\lambda_y) \right] < (\lambda_x + \lambda_y)(M_y - M_x - \lambda_y) < 0 \\ \text{Since plausible rating systems, we expect } M_y \le M_x,\ \text{K} \ge 0 \text{ and } \lambda_y > 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{d}{d\lambda_y} AUROC \le 0$$

3.2 Logistic Distribution

A two parameter Logistic distribution CDF is:

$$F(z) \equiv P(Z \le z) = \frac{e^{\frac{z-M}{\lambda}}}{1+e^{\frac{z-M}{\lambda}}},$$

where $z \in ¡$ and $\lambda > 0$. Here M is a mean parameter.

The inverse CDF of a two parameter Logistic Distribution is:

$$F^{-1}(p) = M + \lambda \ln\left(\frac{p}{1-p}\right),$$

where $p \in [0,1]$. Again assuming y is the *HR(C)* and x is the *FAR(C)*, we have:

$$x = F_X(C) = \frac{e^{\frac{C-M_x}{\lambda_x}}}{1+e^{\frac{C-M_x}{\lambda_x}}} \Rightarrow C = M_x + \lambda_x \ln\left(\frac{x}{1-x}\right) \quad \text{and}$$

$$y = F_Y(C) = F_Y\left(F_X^{-1}(x)\right) = \frac{e^{\frac{C-M_y}{\lambda_y}}}{1+e^{\frac{C-M_y}{\lambda_y}}} = \frac{e^{\frac{M_x-M_y}{\lambda_y}}\left(\frac{x}{1-x}\right)^{\frac{\lambda_x}{\lambda_y}}}{1+e^{\frac{M_x-M_y}{\lambda_y}}\left(\frac{x}{1-x}\right)^{\frac{\lambda_x}{\lambda_y}}}$$

Similarly to the Weibull Distribution case, the AUROC with the above ROC specification can always be evaluated numerically. Moreover, by assuming $\lambda_x = \lambda_y = 1$, and in what follows assume that K does not equal 1, $K = e^{M_x - M_y}$, the above ROC equation can be simplified to

$$y = \frac{e^{M_x-M_y}x}{1-x+e^{M_x-M_y}x} = \frac{Kx}{1+(K-1)x}$$

The AUROC can be now derived analytically.

$$AUROC = K\int_0^1 \frac{x}{1+(K-1)x}dx = \frac{K}{K-1}\int_0^1 \frac{(K-1)x}{1+(K-1)x}dx$$

Let $u = 1+(K-1)x$

$$AUROC = \frac{K}{\left(K-1\right)^2}\int_1^K \left(1-\frac{1}{u}\right)du = \frac{K}{K-1}\left(1-\frac{\ln K}{K-1}\right) \Rightarrow \lim_{K\to\infty} AUROC = 1$$

3.3 Normal Distribution

A two parameter Normal distribution CDF is:

$$F\left(z\right) \equiv P\left(Z \le z\right) = \int_{-\infty}^{z} \frac{1}{\lambda\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{x-M}{\lambda}\right)^2} dx = \Phi\left(\frac{z-M}{\lambda}\right),$$

where $z \in$ ¡ and the $\Phi\left(.\right)$ is the standard Normal probability distribution function.

The inverse CDF of a two parameter Logistic Distribution is: $F^{-1}\left(p\right) = M + \lambda\Phi^{-1}\left(p\right)$,

where $p \in \left[0,1\right]$. For *y* is the *HR(C)* and *x* is the *FAR(C)*, we have:

$$x = \Phi\left(\frac{C-M_x}{\lambda_x}\right) \Rightarrow C = M_x + \lambda_x\Phi^{-1}\left(x\right)$$

$$y = \Phi\left(\frac{C-M_y}{\lambda_y}\right) = \Phi\left(\frac{\left(M_x-M_y\right)+\lambda_x\Phi^{-1}\left(x\right)}{\lambda_y}\right)$$, which gives the ROC curve.

Therefore, $$AUROC = \int_0^1 \Phi\left(\frac{\left(M_x-M_y\right)+\lambda_x\Phi^{-1}\left(x\right)}{\lambda_y}\right)dx$$

This function can easily be evaluated numerically to obtain the AUROC.

Property

AUROC increases with $M_x - M_y$ and in particular if $M_x - M_y = 0$, AUROC = 0.5

Proof and more properties can be found in Appendix B

3.4 Mixed Models

It is obvious that as long as we have parametric distribution families for the defaulters and non-defaulters, we can always calculate an AUROC for the two score samples from equation (1) in section 2, even with two different parametric distributions for the two populations.

4. Performance Evaluation on the AUROC Estimation with Simulated Data

Using simulated data, we carry out performance evaluations on AUROC estimations using both the non-parametric Mann-Whitney Statistic and the analytic approach suggested in this paper.

We first assume some known parametric distributions for the credit scores of defaulters and non-defaulters, and, by doing this, we know the theoretical value of the AUROC. After generating simulated sample data from the assumed distributions for defaulter and non-defaulter scores, we estimate the AUROC and its confidence interval (**CI**) using the two approaches. We repeat the simulation and estimation procedures a number of times. We then compare the accuracy of the AUROC estimation and the CI of the two approaches. Finally, we change the parameter values of the assumed distribution and repeat the simulation. We repeat the above procedures to evaluate the performance of the two approaches subject to different theoretical AUROC index values with different defaulter sample sizes. We choose the following distributions: two-parameter Normal Distributions, one-parameter Exponential Distributions and Weibull Distributions with various shape and scale parameters.

4.1 Performance evaluations under the Normal Distribution Assumption

We assume Normal Distributions for our parametric distribution of the credit scores of both defaulters and non-defaulters. The theoretical value of AUROC for the normal score samples is evaluated numerically[①]. The non-parametric estimate of AUROC is carried out using the ROC module in SPSS and we use the bootstrap to re-sample 1000 replications to obtain the estimates of the analytic approach which also generates a two-sided 95% CI. The parameters of the parametric distribution are estimated for each replication and substituted back into the analytic AUROC formula. We then define the error as the difference between model estimates based on a sample and the theoretical AUROC value, and compare the mean error and mean absolute error for the two approaches. The width of the confidence interval is also compared.

We generate 50 normal samples from six different settings. Settings 1, 2 and 3, consisting of Group 1, target the AUROC at a low value, while settings 4, 5 and 6, Group 2, target the AUROC at a high value. Within each group there are three defaulter sample sizes: 20, 100 and 500. Credit rating models can be applied to at least three different types of groups: credit risk with Corporate, counter party default risk in Trading Books, and credit risk in credit card and other loan type Banking Books. The default sample of Corporate is usually small, such as 50 in ten years, especially under a good economic cycle. Meanwhile, the number of defaults in a loan book or a credit card book in a commercial bank's banking book can be fairly large, usually in excess of several hundreds a year. The reason for selecting different defaulter sample sizes is to assess for which type of problem the analytic approach outperforms the non-parametric approach. We define a performance statistic as follows:

Difference = Non-Parametric Estimate – Analytic Estimate

(Ratio to N) = Difference / (Non-Parametric Estimate).

Normal Setting 1-3:

Normal Distributions			Setting 1	Setting 2	Setting 3
Sample	Mean	Standard deviation	# of Observation	# of Observation	# of Observation
X	2	2	1000	1000	1000
Y	1	3	20	100	500

Theoretical AUROC= 0.609239

Results on estimation error with 50 simulated normal samples & 1000 replication bootstrap

Setting N1

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.007482	0.007466	0.000016	
Mean ABS Error	0.050164	0.048408	0.001756	3.50%
Mean CI Width	0.289225	0.276522	0.012703	4.39%

Setting N2

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.000404	-0.081959	0.082363	
Mean ABS Error	0.025946	0.024957	0.000989	3.81%
Mean CI Width	0.136364	0.130728	0.005636	4.13%

Setting N3

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.002643	0.002752	-0.000109	
Mean ABS Error	0.014172	0.014809	-0.000636	-4.49%
Mean CI Width	0.064965	0.062608	0.002357	3.63%

Normal Setting 4-6:

Normal Distributions			Setting 4	Setting 5	Setting 6
Sample	Mean	Standard deviation	# of Observation	# of Observation	# of Observation
X	2	0.5	1000	1000	1000
Y	1	1	20	100	500

Theoretical AUROC= 0.814448

Results on estimation error with 50 simulated normal samples & 1000 replication bootstrap

Setting N4

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.009138	0.006718	0.002421	
Mean ABS Error	0.046588	0.045725	0.000863	1.85%
Mean CI Width	0.232187	0.215922	0.016265	7.01%

Setting N5

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.001187	0.000510	0.000678	
Mean ABS Error	0.025564	0.024311	0.001253	4.90%
Mean CI Width	0.112444	0.107148	0.005296	4.71%

Setting N6

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.001470	0.001303	0.000167	
Mean ABS Error	0.012239	0.011061	0.001178	9.62%
Mean CI Width	0.052653	0.049464	0.003189	6.06%

In tables N1-N6, all mean confidence interval widths show that the estimates of the analytic approach are better than the non-parametric estimates. As for the mean error and the mean absolute error, analytic estimates outperform the non-parametric estimates in tables N1, N2 and N4- N6. (Ratio to N) shows the percentage difference from the non-parametric approach estimate. The larger the (Ratio to N), the more the analytic approach outperforms the non-parametric approach.

4.2 Performance evaluations under Exponential Distribution Assumption

In this performance evaluation we assume Exponential Distributions for our parametric distribution of the credit scores of both the defaulters and the non-defaulters. The theoretical value of AUROC for the Exponential score samples is evaluated analytically by the closed form formula (3) in section 3.1. The performance evaluation setting is very similar to that with Normal Distribution. Again there are 6 settings across different AUROC values and defaulter sample sizes.

Exponential Setting 1-3:

Exponential Distributions		Setting 1	Setting 2	Setting 3
Sample	Scale Parameter (Lamda)	# of Observation	# of Observation	# of Observation
X	3	1000	1000	1000
Y	1.5	20	100	500

Theoretical AUROC=0.666667

Results on estimation error with 50 simulated normal samples & 1000 replication bootstrap

Setting E1

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	-0.008179	-0.007035	-0.001144	
Mean ABS Error	0.040993	0.040056	0.000938	2.29%
Mean CI Width	0.209540	0.189586	0.019954	9.52%

Setting E2

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	-0.000987	0.000034	-0.001021	
Mean ABS Error	0.025320	0.021922	0.003398	13.42%
Mean CI Width	0.099043	0.088280	0.010763	10.87%

Setting E3

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	-0.002926	-0.003401	0.000475	
Mean ABS Error	0.011471	0.011015	0.000456	3.98%
Mean CI Width	0.055636	0.047672	0.007964	14.31%

Exponential Setting 4-6:

Exponential Distributions		Setting 4	Setting 5	Setting 6
Sample	Scale Parameter (Lambda)	# of Observation	# of Observation	# of Observation
X	4	1000	1000	1000
Y	1	20	100	500

Theoretical AUROC=0.800000

Results on estimation error with 50 simulated normal samples & 1000 replication bootstrap

Setting E4

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	-0.008576	-0.006721	-0.001855	
Mean ABS Error	0.033790	0.031174	0.002616	7.74%
Mean CI Width	0.145500	0.132758	0.012742	8.76%

Setting E5

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.002783	0.003403	-0.000621	
Mean ABS Error	0.015655	0.014320	0.001335	8.53%
Mean CI Width	0.071140	0.064132	0.007008	9.85%

Setting E6

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.000118	0.000521	-0.000403	
Mean ABS Error	0.007710	0.007495	0.000215	2.79%
Mean CI Width	0.043742	0.034280	0.009462	21.63%

In table E1-E6, all the mean absolute error and the mean confidence interval widths show that the estimates of the analytic approach are better than the non-parametric estimates. (Ratio to N) shows that the non-parametric approach estimates provide a significantly better confidence interval than the non-parametric estimates.

4.3 Performance evaluations under Weibull Distribution Assumption

In this performance evaluation we assume Weibull Distributions with scale and shape parameters for our parametric distribution of the credit scores of both the defaulters and the non-defaulters. The theoretical value of AUROC for the Weibull score samples is evaluated analytically by the closed form formula (2) in section 3.1 by setting the location parameters to zero. The maximum estimation of sample distribution parameters are obtained by a numerical approximation. Since we have a shape parameter for the Weibull Distribution which may shift the shape of the distribution significantly, we evaluate the performance of the two approaches under two cases: with the same shape parameter for defaulter and non-defaulter sample, and with different shape parameters. The theoretical value of AUROC for the normal score samples is also evaluated numerically[2]. The rest of the performance evaluation setting is very similar to that of the Normal Distribution. Here also, there are six settings across different AUROC values and defaulter sample sizes.

Weibull Setting 1-3:

Weibull Distributions			Setting 1	Setting 2	Setting 3
Sample	Shape parameter	Scale	# of Observation	# of Observation	# of Observation
X	2	2	1000	1000	1000
Y	1	1	20	100	500

Theoretical AUROC= 0.757867

Results on estimation error with 50 simulated normal samples & 1000 replication bootstrap

Setting W1

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.005128	0.010230	-0.005102	
Mean ABS Error	0.051701	0.054179	-0.002478	-4.79%
Mean CI Width	0.242836	0.226842	0.015994	6.59%

Setting W2

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.001110	0.000983	0.000127	
Mean ABS Error	0.022661	0.022363	0.000298	1.32%
Mean CI Width	0.112448	0.109910	0.002538	2.26%

Setting W3

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.0027541	0.0030963	-0.000342	
Mean ABS Error	0.0123445	0.0118544	0.000490	3.97%
Mean CI Width	0.0548159	0.0533400	0.001476	2.69%

Weibull Setting 4-6:

Weibull Distributions			Setting 4	Setting 5	Setting 6
Sample	Shape parameter	Scale	# of Observation	# of Observation	# of Observation
X	1	3	1000	1000	1000
Y	1	1	20	100	500

Theoretical AUROC= 0.75

Results on estimation error with 50 simulated normal samples & 1000 replication bootstrap

Setting W4

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.000084	0.000314	-0.000231	
Mean ABS Error	0.035960	0.036155	-0.000195	-0.54%
Mean CI Width	0.168248	0.165242	0.003006	1.79%

Setting W5

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.003680	0.003795	-0.000115	
Mean ABS Error	0.018331	0.017988	0.000343	1.87%
Mean CI Width	0.082652	0.081830	0.000822	0.99%

Setting W6

Approach	Non-parametric	Analytic	Difference	Ratio to N
Mean Error	0.003889	0.003961	-0.000072	
Mean ABS Error	0.009632	0.009525	0.005340	1.11%
Mean CI Width	0.048446	0.047586	0.000860	1.77%

In tables W1-W6, all mean confidence interval widths show that the estimates of the analytic approach are marginally better than the non-parametric estimates. As for the mean error and the mean absolute error, analytic estimates marginally outperform the non-parametric estimates in tables W2, W3, W5 and W6. Because we use numerical approximation for sample maximum likelihood estimates and because the estimation error could be fairly large when we have a small sample, we observe that this estimation error is passed through our analytic estimation for the AUROC index making the mean absolute errors estimated from the analytic approach larger than the non-parametric approach in setting W1 and W4. This also reduces the gain of the analytic approach over the non-parametric approach when compared with the previous tests.

Summary:

Although the analytic approach gives no better estimates than the non-parametric one when we use approximated maximum likelihood estimates for small samples, the performance evaluation shows that the analytic approach works at least as well as the non-parametric approach in the above tests and, in most cases, provides better mean absolute error estimates and confidence interval estimates.

The above discussion has the following implications. If appropriate parametric distributions for the defaulter and non-defaulter scores can be identified, then the AUROC and its confidence interval can be estimated more accurately using the analytic approach. On the other hand, if the rating model can be designed so that the score sample is generated by some specific parametric distribution families, then a better rating model could be found by using the analytic AUROC as the objective function to maximize in the model selecting process.

Another interesting finding is the effect of defaulter sample size on AUROC. The above experiments clearly show the level of estimation error in both methods with different sample sizes, and the error can be substantially large if we only have a small defaulter sample.

In addition, although it is not very clear, from the results in section 4.1 and 4.2, the analytic approach seems to provide more gain over the non-parametric approach when the AUROC index is in its high value region than in its low value region. The reason for this is not clear so more research is needed.

5 Conclusions

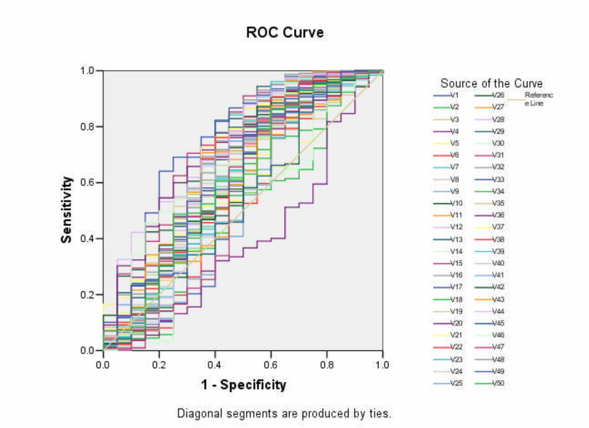
This paper reviews some of the prevailing credit rating model validation approaches and, in particular, studies the analytic properties of the ROC curve and its summary index AUROC. We use the concept of the population ROC curve to build analytic models of ROC curves. It has been shown through simulation studies that greater accuracy of estimation relative to the non-parametric methods can be achieved. We also show that there are some situations where the accuracy gain of the analytic ROC model may decrease, a finding that should be taken into account when applying the analytic models to practical applications. In addition, it should be noted that unless the rating score distribution forms is known, necessary Exploratory Data Analysis should be carried out before using the analytic AUROC approach to minimize the risk of distribution misspecification.

Moreover, with some distributions, where the closed form solution of AUROC is available, analytic AUROC can be directly used as an objective function to maximize during the rating model selection procedure. This means that if the rating scores can be transformed into those distributions, analytic AUROC could offer a powerful model selection tool.

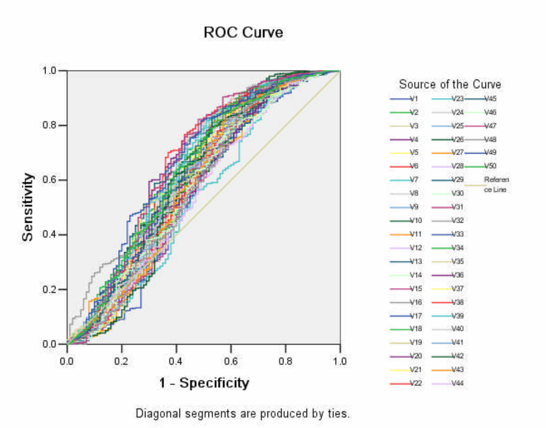
Finally, we also studied the performance of both non-parametric and analytic ROC models under different defaulter sample size, research that had not been done previously. The error size can be substantially significant when we have a small defaulter sample, a frequently met situation in corporate credit risk studies and in periods of benign macro-economic background.

Appendix A1: Non-parametric ROC curve for Normally Distributed Samples

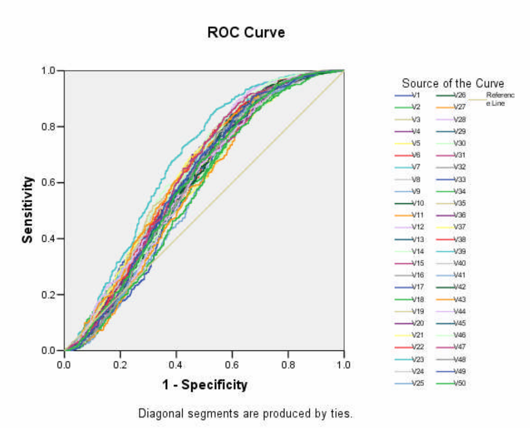
Non-parametric ROC curve under setting 1:



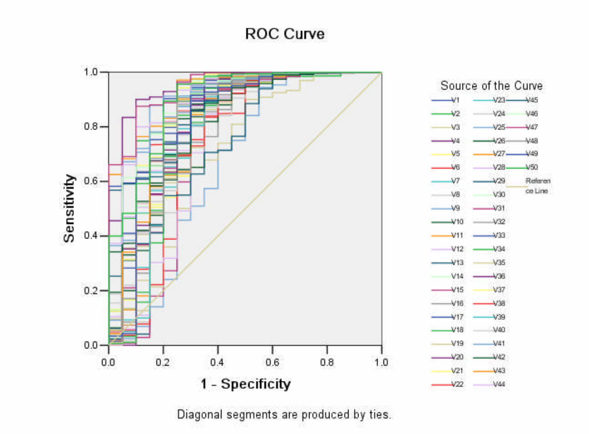
Non-parametric ROC curve under setting 2:



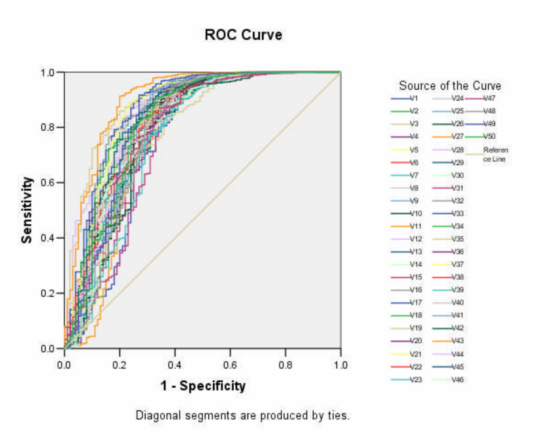
Non-parametric ROC curve under setting 3:



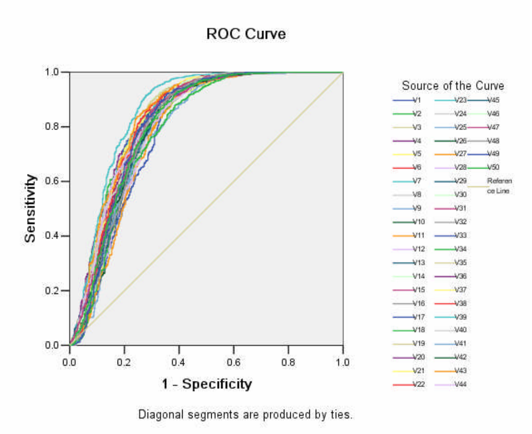
Non-parametric ROC curve under setting 4:



Non-parametric ROC curve under setting 5:



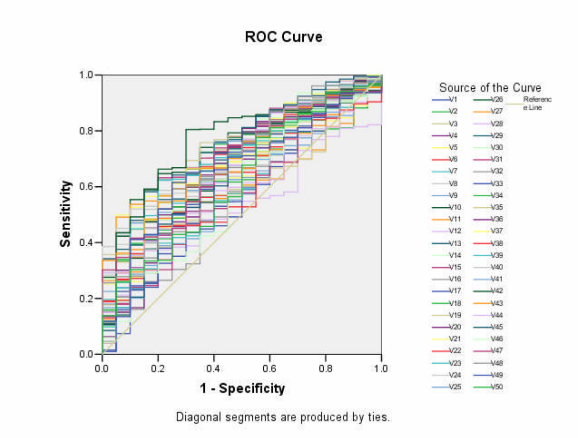
Non-parametric ROC curve under setting 6:



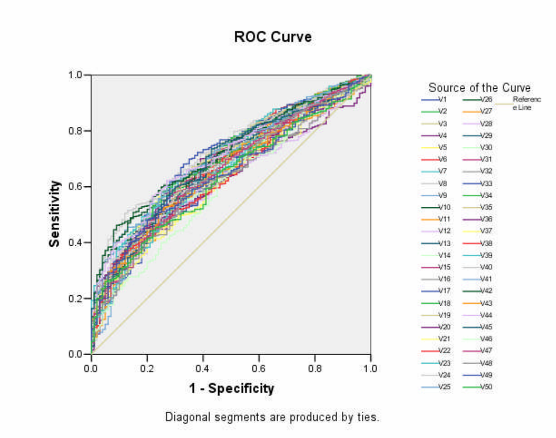
For example, ROC curve V*i* is plotted using the data of simulated sample number *i* generated under a specified setting.

Appendix A2: Non-parametric ROC curve for Exponentially Distributed Samples

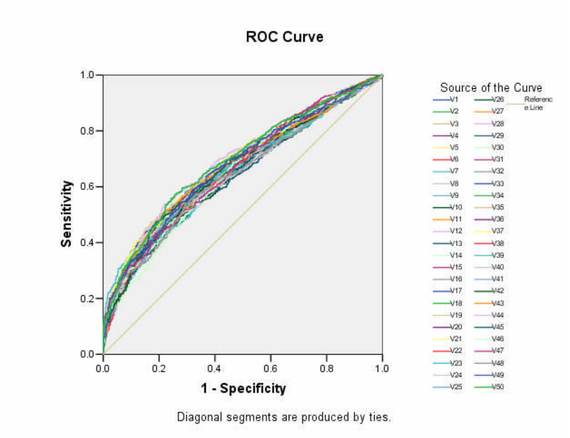
Non-parametric ROC curve under setting 1:



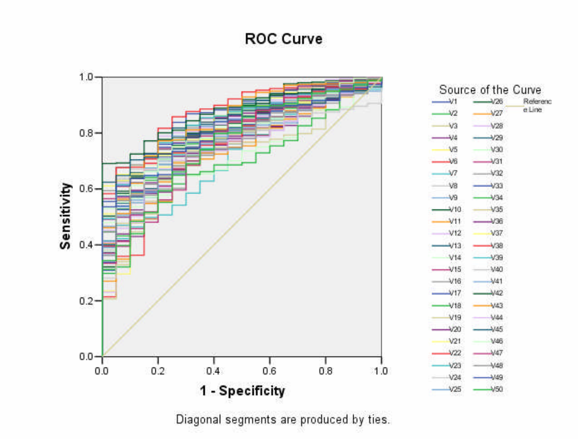
Non-parametric ROC curve under setting 2:



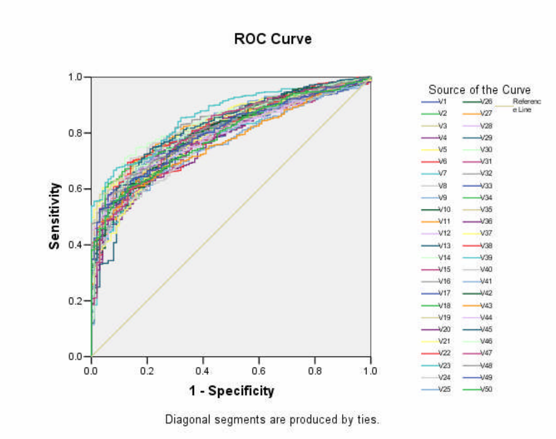
Non-parametric ROC curve under setting 3:



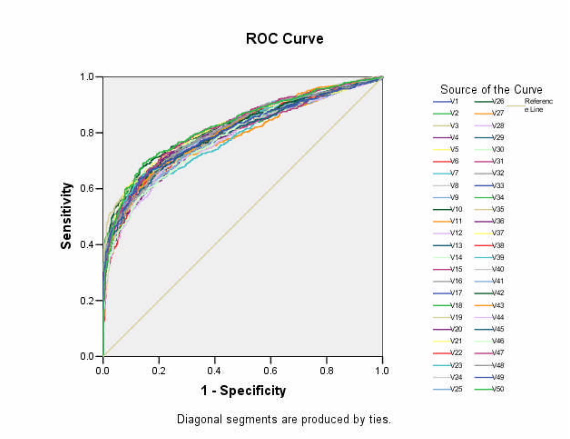
Non-parametric ROC curve under setting 4:



Non-parametric ROC curve under setting 5:



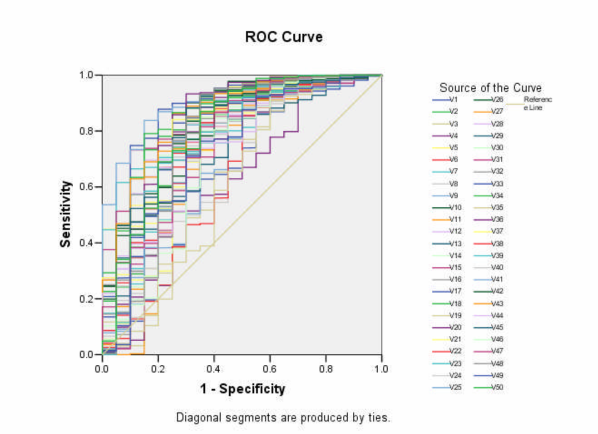
Non-parametric ROC curve under setting 6:



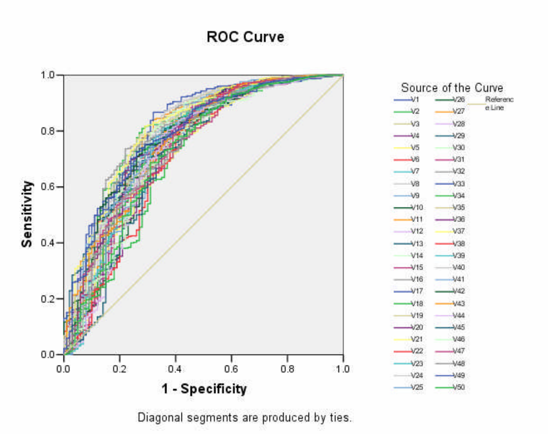
For example, ROC curve V*i* is plotted using the data of simulated sample number *i* generated under a specified setting.

Appendix A3: Non-parametric ROC curve for Weibull Distributed Samples

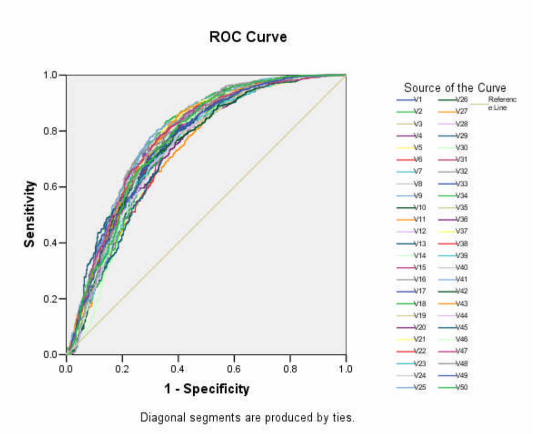
Non-parametric ROC curve under setting 1:



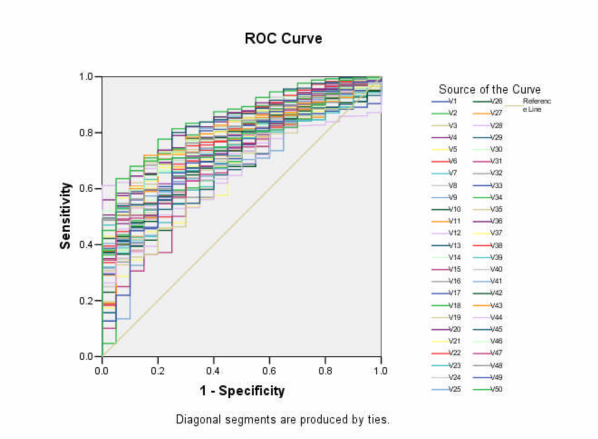
Non-parametric ROC curve under setting 2:



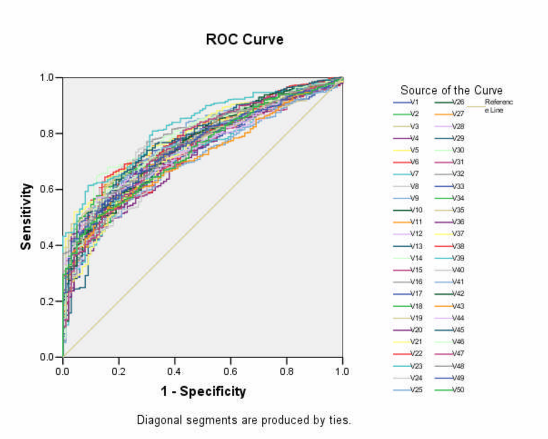
Non-parametric ROC curve under setting 3:



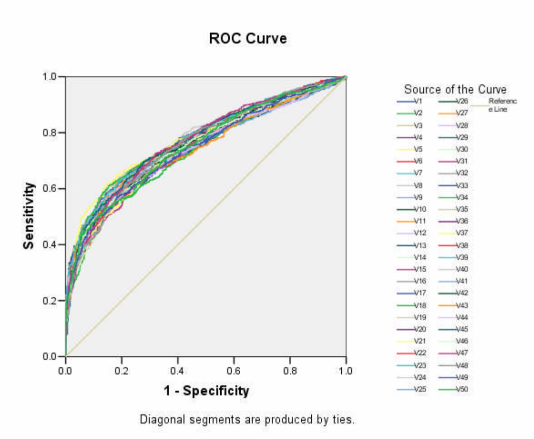
Non-parametric ROC curve under setting 4:



Non-parametric ROC curve under setting 5:



Non-parametric ROC curve under setting 6:



For example, ROC curve V*i* is plotted using the data of simulated sample number *i* generated under a specified setting.

Appendix B: The properties of AUROC for Normally Distributed Sample

Property 1:

AUROC increases with $M_x - M_y$ and in particular if $M_x - M_y = 0$, AUROC = 0.5

Proof:

For inverse normal distribution function $u = \Phi^{-1}(v)$, $v \in [0,1]$ and $u \in (-\infty, +\infty)$. It is an odd function in orthogonal coordinates with centre of (v = 0.5, u = 0).

For cumulative normal distribution function $t = \Phi(u)$. This is also an odd function in orthogonal coordinates with centre of (u=0, t =0.5).

It follows that $f(x) = \Phi\left(\frac{\lambda_x}{\lambda_y}\Phi^{-1}(x)\right)$ is also an odd function in orthogonal coordinates with centre of (x = 0.5, $f(x)$ = 0.5), when $M_x - M_y = 0$. Rewrite $f(x)$ as following:

$f(x) = [f(x) - 0.5] + 0.5 = g(x) + 0.5$, where $g(x)$ is an odd function with centre of (x = 0.5, $g(x)$ = 0). Then we can show that

$$\text{AUROC}=\int_0^1 f(x)dx = \int_0^1 g(x)dx + \int_0^1 0.5dx = \int_0^{0.5} g(x)dx + \int_{0.5}^1 g(x)dx + \int_0^1 0.5dx$$
$$= \int_0^{0.5} g(x)dx - \int_0^{0.5} g(x)dx + \int_0^1 0.5dx = \int_0^1 0.5dx = 0.5 \qquad \text{QED}$$

The above property is also quite intuitive. If the means of two normally distributed populations equal each other, then overall there is no discriminatory power of the models based on this rating mechanism, i.e. neither X or Y FSD. So the AUROC is 0.5. A special case for this is when we have two identical distributions for X and Y. Therefore, Second Order Stochastic Dominance (SSD) cannot be identified by AUROC, when $M_x - M_y = 0$.

Property 2

The relations with λ_x and λ_y are slightly more complicated.

$$AUROC \begin{cases} \in (0.5,\ 1), & \text{decreases with } \lambda_x, \text{ when } M_x - M_y > 0 \\ = 0.5, & \text{irrelavent to } \lambda_x, \quad \text{when } M_x - M_y = 0 \\ \in (0,\ 0.5), & \text{increases with } \lambda_x, \text{ when } M_x - M_y < 0 \end{cases}$$

$$AUROC \begin{cases} \in (0.5,\ 1), & \text{decreases with } \lambda_y, \text{ when } M_x - M_y > 0 \\ = 0.5, & \text{irrelavent to } \lambda_y, \quad \text{when } M_x - M_y = 0 \\ \in (0,\ 0.5), & \text{increases with } \lambda_y, \text{ when } M_x - M_y < 0 \end{cases}$$

We are only interested in the rating models and this is the case where X should FSD Y, i.e., $M_x - M_y > 0$, so it is clear that with smaller standards of the two normal distributions, the two samples are more separated than those with larger standard deviations when $M_x - M_y > 0$.

The graphs below shows the AUROC with different Lambda settings. Lambda of X is written as L.X and Lambda of Y is L.Y

Figure 1: Normal Distributed AUROC with the same mean

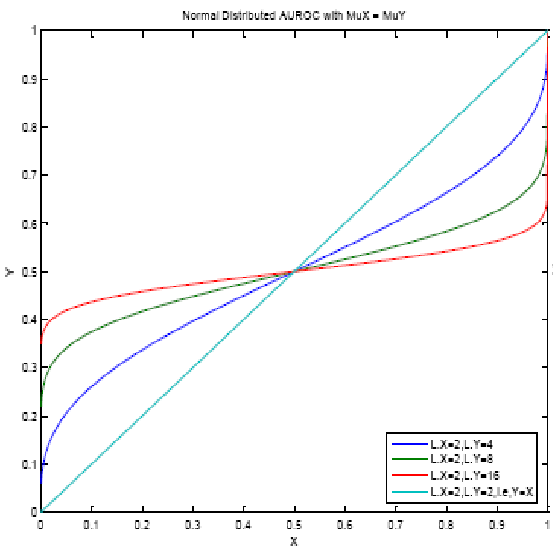
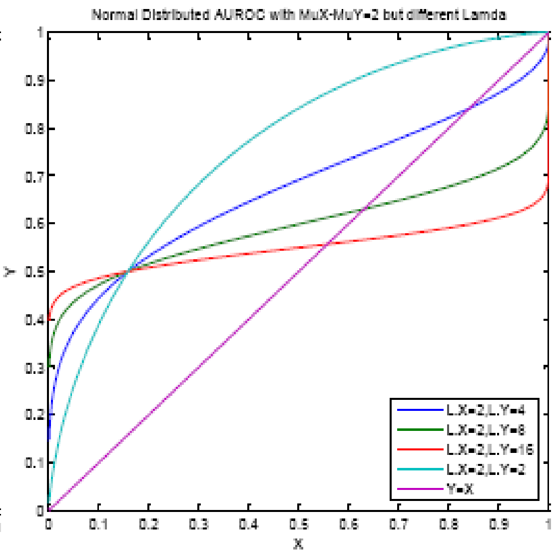


Figure 2: Normal Distributed AUROC with different means



Remark:

The closer the AUROC of a rating system is to 0.5, the less discriminatory power it has. The closer the AUROC of a rating system is to 0 or 1, the better its discriminatory power. Therefore, under the Normally Distributed scoring variable assumption, the smaller the variance, the better the discriminatory power the rating system has.

When $M_x - M_y < 0$, a scoring system would give defaulters, *Y*, higher scores. Hence even the discriminatory power is higher when we have smaller variances on *X* and *Y* in this case, but the AUROC will be smaller.

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① The theoretical AUROC is approximated by 100000 partitions, while the bootstrap estimation is approximated by 10000 partitions.

② The theoretical AUROC is approximated by 100000 partitions, while the bootstrap estimation is approximated by 10000 partitions.