Euler Allocation: Theory and Practice

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Abstract

Despite the fact that the Euler allocation principle has been adopted by many financial institutions for their internal capital allocation process, a comprehensive description of Euler allocation seems still to be missing. We try to fill this gap by presenting the theoretical background as well as practical aspects. In particular, we discuss how Euler risk contributions can be estimated for some important risk measures.

1 Introduction

In many financial institutions, there is a well established practice of measuring the risk of their portfolios in terms of *economic capital* (cf., e.g. Dev, 2004). Measuring portfolio-wide economic capital, however, is only the first step towards active, portfolio-oriented risk management. For purposes like identification of concentrations, risk-sensitive pricing or portfolio optimization it is also necessary to decompose portfolio-wide economic capital into a sum of risk contributions by sub-portfolios or single exposures (see, e.g., Litterman, 1996).

Overviews of a variety of different methodologies for this so-called *capital allocation* were given, e.g., by Koyluoglu and Stoker (2002) and Urban et al. (2004). McNeil et al. (2005, Section 6.3) discuss in some detail the *Euler allocation* principle. The goal with this paper is to provide more background information on the Euler allocation, in particular with respect to the connection of Euler's theorem and risk diversification and some estimation issues with Euler risk contributions. The presentation is largely based on work on the subject by the author but, of course, refers to other authors where appropriate.

Section 2 "Theory" is mainly devoted to a motivation of the Euler allocation principle by taking recourse to the economic concept of RORAC compatibility (Section 2.2). Additionally, in Section 2.4, it is demonstrated that Euler risk contributions are well-suited as a tool for the detection of risk concentrations. Section 3 "Practice" presents the formulae that are needed to calculate Euler risk contributions for standard deviation based risk measures, Value-at-Risk (VaR) and Expected Shortfall (ES). Some VaR-specific estimation issues are discussed in Section 3.2. Furthermore, in Section 3.4, it is shown that there is a quite natural relationship between Euler contributions to VaR and the Nadaraya-Watson kernel estimator for conditional expectations. The paper concludes with a brief summary in Section 4. Appendix A provides some useful facts on homogeneous functions.

2 Theory

The Euler allocation principle may be applied to any risk measure that is homogeneous of degree 1 in the sense of Definition A.1 and differentiable in an appropriate sense. After having introduced the basic

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setting for the paper in Section 2.1, we discuss in Section 2.2 the economic motivation for the use of Euler risk contributions. The fact that the Euler allocation principle can be derived by economic considerations constitutes the – maybe – most appealing feature of this principle. We show in Sections 2.3 and 2.4 that the economic foundation of the Euler allocation becomes even stronger in the case of sub-additive risk measures because it can then be used for efficient detection of risk concentration.

2.1 Basic setting

Suppose that real-valued random variables $X_1, \ldots, X_n$ are given that stand for the profits and losses with the assets (or some sub-portfolios) in a portfolio. Let X denote the portfolio-wide profit/loss, i.e. let

$$X \;=\; \sum_{i=1}^{n} X_i. \tag{2.1}$$

The economic capital (EC) required by the portfolio (i.e. capital as a buffer against high losses caused by the portfolio) is determined with a *risk measure* ρ, i.e.

$$\mathrm{EC} \;=\; \rho(X). \tag{2.2}$$

In practice, usually, ρ is related to the variance or a quantile of the portfolio loss distribution. See Section 3 for some examples of how ρ can be chosen.

For some kinds of risk (in particular credit risk), for risk management traditionally only losses are considered. Let $L_i \geq 0$ denote the loss with (credit) asset i and assume that $L = \sum L_i$ stands for portfolio-wide loss. Then this loss perspective on portfolio risk can be reconciled with the profit/loss perspective from (2.1) and (2.2) by considering

$$X_i \;=\; g_i - L_i, \tag{2.3}$$

where g_i denotes the lender's stipulated gain with credit asset i if the credit is repaid regularly. In the following, as a general rule we will adopt the profit/loss perspective from (2.2).

It is useful to allow for some dynamics in model (2.1) by introducing variables $u = (u_1, \ldots, u_n)$:

$$X(u) \;=\; X(u_1, \ldots, u_n) \;=\; \sum_{i=1}^{n} u_i\, X_i. \tag{2.4}$$

Then we have obviously $X = X(1, \ldots, 1)$. The variable u_i can be interpreted as amount of money invested in the asset which underlies X_i or just as the credit exposure if X_i is chosen as a default indicator. For the purpose of this paper we assume that the probability distribution of the random vector $(X_1, \ldots, X_n)$ is fixed. We will, however, consider some variations of the variable u. It is then convenient to introduce the function

$$f_{\rho,X}(u) \;=\; \rho(X(u)). \tag{2.5}$$

For the same risk measure ρ, the function $f_{\rho,X}$ can look quite different for different distributions of X. As we assume that the distribution of X is fixed, we can nevertheless drop the index X and write f_ρ for $f_{\rho,X}$.

In this paper, we focus attention to (positively) *homogeneous* risk measures ρ and functions f_ρ (see Appendix A for some important properties of such functions).

2.2 Defining risk contributions

When portfolio-wide economic capital is measured according to (2.2), it may be useful to answer the question: How much contributes asset (or sub-portfolio) i to $\mathrm{EC} = \rho(X)$? Some potential applications of the answer to this question will be given below. For the time being, we denote the (still to be defined) *risk contribution* of X_i to $\rho(X)$ by $\rho(X_i \,|\, X)$.

Definition 2.1 *Let $\mu_i = \mathrm{E}[X_i]$. Then*

- *the total portfolio* <u>R</u>eturn <u>o</u>n <u>R</u>isk <u>A</u>djusted <u>C</u>apital[1] *is defined by*

$$\mathrm{RORAC}(X) \;=\; \frac{\mathrm{E}[X]}{\rho(X)} \;=\; \frac{\sum_{i=1}^m \mu_i}{\rho(X)},$$

- *the portfolio-related* RORAC *of the i-th asset is defined by*

$$\mathrm{RORAC}(X_i\,|\,X) \;=\; \frac{\mathrm{E}[X_i]}{\rho(X_i\,|\,X)} \;=\; \frac{\mu_i}{\rho(X_i\,|\,X)}.$$

Based on the notion of RORAC as introduced in Definition 2.1, two properties of risk contributions can be stated that are desirable from an economic point of view.

Definition 2.2 *Let X denote portfolio-wide profit/loss as in (2.1).*

- *Risk contributions $\rho(X_1\,|\,X), \ldots, \rho(X_n\,|\,X)$ to portfolio-wide risk $\rho(X)$ satisfy the* full allocation *property if*

$$\sum_{i=1}^n \rho(X_i\,|\,X) \;=\; \rho(X).$$

- *Risk contributions $\rho(X_i\,|\,X)$ are* RORAC compatible *if there are some $\epsilon_i > 0$ such that*

$$\mathrm{RORAC}(X_i\,|\,X) \;>\; \mathrm{RORAC}(X) \qquad \Rightarrow \qquad \mathrm{RORAC}(X + h\,X_i) \;>\; \mathrm{RORAC}(X)$$

for all $0 < h < \epsilon_i$.

It turns out that in the case of a "smooth" risk measure ρ, requiring the RORAC compatibility property of Definition 2.2 completely determines the risk contributions $\rho(X_i\,|\,X)$.

Proposition 2.1 *Let ρ be a risk measure and f_ρ be the function that corresponds to ρ according to (2.4) and (2.5). Assume that f_ρ is continuously differentiable. If there are risk contributions $\rho(X_1\,|\,X), \ldots,$ $\rho(X_n\,|\,X)$ that are RORAC compatible in the sense of Definition 2.2 for arbitrary expected values $\mu_1, \ldots, \mu_n$ of $X_1, \ldots, X_n$, then $\rho(X_i\,|\,X)$ is uniquely determined as*

$$\rho_{\mathrm{Euler}}(X_i\,|\,X) \;=\; \frac{d\,\rho}{d\,h}(X + h\,X_i)\big|_{h=0} \;=\; \frac{\partial\, f_\rho}{\partial\, u_i}(1, \ldots, 1). \tag{2.6}$$

See Theorem 4.4 of Tasche (1999) for a proof of Proposition 2.1. It is easy to see that risk contributions defined by (2.6) are always RORAC compatible.

What about the full allocation property of Definition 2.2? Assume that the function f_ρ corresponding to the risk measure ρ is continuously differentiable. Then, by Euler's theorem on homogeneous functions (see Theorem A.1 in Appendix A), f_ρ satisfies the equation

$$f_\rho(u) \;= \sum_{i=1}^n u_i\, \frac{\partial\, f_\rho(u)}{\partial\, u_i} \tag{2.7}$$

for all u in its range of definition if and only if it is homogeneous of degree 1 (cf. Definition A.1). Consequently, for the risk contributions to risk measures ρ with continuously differentiable f_ρ the two properties of Definition 2.2 can obtain at the same time if and only if the risk measure is homogeneous of degree 1. The risk contributions are then uniquely determined by (2.6).

[1] Note that performance measurement by RORAC can be motivated by Markowitz-type risk-return optimisation for general risk measures that are homogeneous of any degree τ (see Tasche, 1999, Section 6).

Remark 2.1 *If ρ is a risk measure which is homogeneous of degree 1 (in the sense of Definition A.1), then risk contributions according to (2.6) are called* Euler contributions. *Euler contributions satisfy both properties of Definition 2.2, i.e. they are RORAC compatible and add up to portfolio-wide risk. The process of assigning capital to assets or sub-portfolios by calculating Euler contributions is called* Euler allocation.

The use of the Euler allocation principle was justified by several authors with different reasonings:

- Patrik et al. (1999) argued from a practitioner's view emphasizing mainly the fact that the risk contributions according to the Euler principle by (2.6) naturally add up to the portfolio-wide economic capital.
- Litterman (1996) and Tasche (1999, as shown above) pointed out that the Euler principle is fully compatible with economically sensible portfolio diagnostics and optimization.
- Denault (2001) derived the Euler principle by game-theoretic considerations.
- In the context of capital allocation for insurance companies, Myers and Read (2001) argued that applying the Euler principle to the expected "default value" (essentially $\mathrm{E}[\max(X, 0)]$) of the insurance portfolio is most appropriate for deriving line-by-line surplus requirements.
- Kalkbrener (2005) presented an axiomatic approach to capital allocation and risk contributions. One of his axioms requires that risk contributions do not exceed the corresponding stand-alone risks. From this axiom in connection with more technical conditions, in the context of sub-additive and positively homogeneous risk measures, Kalkbrener concluded that the Euler principle is the only allocation principle to be compatible with the "diversification"-axiom (see also Kalkbrener et al., 2004; Tasche, 2002; and Section 2.3 below).
- More recently, the Euler allocation was criticized for not being compatible with the decentralized risk management functions of large financial institutions (Schwaiger, 2006). Gründl and Schmeiser (2007) even find that capital allocation is not needed at all for insurance companies.

2.3 Contributions to sub-additive risk measures

As risk hedging by diversification plays a major role for portfolio management, we briefly recall the observations by Kalkbrener (2005) and Tasche (1999) on the relation between the Euler allocation principle and diversification.

It is quite common to associate risk measures that reward portfolio diversification with the so-called sub-additivity property (Artzner et al., 1999). A risk measure ρ is *sub-additive* if it satisfies

$$\rho(X + Y) \;\leq\; \rho(X) + \rho(Y) \tag{2.8}$$

for any random variables X, Y in its range of definition. Assume the setting of Section 2.1 and that ρ is a risk measure that is both homogeneous of degree 1 and sub-additive. By Corollary A.1, then the function f_ρ that corresponds to ρ via (2.5) fulfills the equation

$$\sum_{i=1}^{n} u_i\, \frac{\partial\, f_\rho(u+v)}{\partial\, u_i} \;\leq\; f_\rho(u), \quad i = 1, \ldots, n. \tag{2.9a}$$

With $u = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 at i-th position) and $v = (1, \ldots, 1) - u$, (2.9a) implies

$$\rho_{\mathrm{Euler}}(X_i \,|\, X) \;\leq\; \rho(X_i), \quad i = 1, \ldots, n, \tag{2.9b}$$

where $\rho_{\mathrm{Euler}}(X_i \,|\, X)$ is defined by (2.6). Hence, if risk contributions to a homogeneous and sub-additive risk measure are calculated as Euler contributions, then the contributions of single assets will never exceed the assets' stand-alone risks. In particular, risk contributions of credit assets cannot become larger than the face values of the assets.

Actually, Corollary A.1 shows that, for continuously differentiable and risk measures ρ homogeneous of degree 1, property (2.9b) for the Euler contributions and sub-additivity of the risk measure are equivalent. This is of particular relevance for credit risk portfolios where violations of the sub-additivity property are rather observed as violations of (2.9b) than as violations of (2.8) (see Kalkbrener et al., 2004).

Recall the notion of the so-called *marginal risk contribution*[2] for determining the capital required by an individual business, asset, or sub-portfolio. Formally, the marginal risk contribution $\rho_{\mathrm{marg}}(X_i \,|\, X)$ of asset i, $i = 1, \ldots, n$, is defined by

$$\rho_{\mathrm{marg}}(X_i \,|\, X) \;=\; \rho(X) - \rho(X - X_i), \tag{2.10}$$

i.e. by the difference of the portfolio risk with asset i and the portfolio risk without asset i. In the case of continuously differentiable and sub-additive risk measures that are homogeneous of degree 1, it can be shown that marginal risk contributions are always smaller than the corresponding Euler contributions (Tasche, 2004b, Proposition 2).

Proposition 2.2 *Let ρ be a sub-additive and continuously differentiable risk measure that is homogeneous of degree 1. Then the marginal risk contributions $\rho_{\mathrm{marg}}(X_i \,|\, X)$ as defined by (2.10) are smaller than the corresponding Euler contributions, i.e.*

$$\rho_{\mathrm{marg}}(X_i \,|\, X) \;\leq\; \rho_{\mathrm{Euler}}(X_i \,|\, X). \tag{2.11a}$$

In particular, the sum of the marginal risk contributions underestimates total risk, i.e.

$$\sum_{i=1}^{n} \rho_{\mathrm{marg}}(X_i \,|\, X) \;=\; \sum_{i=1}^{n} \big(\rho(X) - \rho(X - X_i)\big) \;\leq\; \rho(X). \tag{2.11b}$$

As a work-around for the problem that marginal risk contributions do not satisfy the full allocation property, sometimes marginal risk contributions are defined as

$$\rho^*_{\mathrm{marg}}(X_i \,|\, X) \;=\; \frac{\rho_{\mathrm{marg}}(X_i \,|\, X)}{\sum_{j=1}^{n} \rho_{\mathrm{marg}}(X_j \,|\, X)}\, \rho(X). \tag{2.12}$$

This way, equality in (2.11b) is forced. In general, however, marginal risk contributions according to (2.12) do not fulfil the RORAC compatibility property from Definition 2.2.

2.4 Measuring concentration and diversification

In BCBS (2006, paragraph 770) the Basel Committee on Banking Supervision states: "A risk concentration is any single exposure or group of exposures with the potential to produce losses large enough (relative to a bank's capital, total assets, or overall risk level) to threaten a bank's health or ability to maintain its core operations. Risk concentrations are arguably the single most important cause of major problems in banks." In BCBS (2006, paragraph 774) the Basel Committee then explains: "A bank's framework for managing credit risk concentrations should be clearly documented and should include a definition of the credit risk concentrations relevant to the bank and how these concentrations and their corresponding limits are calculated." We demonstrate in this section that the Euler allocation as introduced in Section 2.2 is particularly well suited for calculating concentrations.

The concept of concentration index (following Tasche, 2006) we will introduce is based on the idea that the actual risk of a portfolio should be compared to an appropriate worst-case risk of the portfolio in order to be able to identify risk concentration. It turns out that "worst-case risk" can be adequately

[2] This methodology is also called *with-without principle* by some authors.

expressed as maximum dependence of the random variables the portfolio model is based on. In actuarial science, the concept of co-monotonicity is well-known as it supports easy and reasonably conservative representations of dependence structures (see, e.g., Dhaene et al., 2006). Random variables V and W are called co-monotonic if they can be represented as non-decreasing functions of a third random variable Z, i.e.

$$V = h_V(Z) \quad \text{and} \quad W = h_W(Z) \tag{2.13a}$$

for some non-decreasing functions h_V, h_W. As co-monotonicity is implied if V and W are correlated with correlation coefficient 1, it generalizes the concept of linear dependence. A risk measure ρ is called *co-monotonic additive* if for any co-monotonic random variables V and W

$$\rho(V + W) \;=\; \rho(V) + \rho(W). \tag{2.13b}$$

Thus co-monotonic additivity can be interpreted as a specification of worst case scenarios when risk is measured by a sub-additive (see (2.8)) risk measure: nothing worse can occur than co-monotonic random variables – which seems quite natural[3]. These observations suggest the first part of the following definition.

Definition 2.3 *Let $X_1, \ldots, X_n$ be real-valued random variables and let $X = \sum_{i=1}^n X_i$. If ρ is a risk measure such that $\rho(X), \rho(X_1), \ldots, \rho(X_n)$ are defined, then*

$$\mathrm{DI}_\rho(X) \;=\; \frac{\rho(X)}{\sum_{i=1}^n \rho(X_i)} \tag{2.14a}$$

denotes the diversification index *of portfolio X with respect to the risk measure ρ.*
If Euler risk contributions of X_i to $\rho(X)$ in the sense of Remark 2.1 exist, then the ratio

$$\mathrm{DI}_\rho(X_i \,|\, X) \;=\; \frac{\rho_{\mathrm{Euler}}(X_i \,|\, X)}{\rho(X_i)} \tag{2.14b}$$

with $\rho_{\mathrm{Euler}}(X_i \,|\, X)$ being defined by (2.6) denotes the marginal diversification indices *of sub-portfolio X_i with respect to the risk measure ρ.*

Note that without calling the concept “diversification index”, Memmel and Wehn (2006) calculate a diversification index for the German supervisor’s market price risk portfolio. Garcia Cespedes et al. (2006) use the diversification indices as defined here for a representation of portfolio risk as a “diversification index”-weighted sum of stand-alone risks.

Remark 2.2 *Definition 2.3 is most useful when the risk measure ρ under consideration is homogeneous of degree 1, sub-additive, and co-monotonic additive. Additionally, the function f_ρ associated with ρ via (2.7) should be continuously differentiable. Expected shortfall, as considered in Section 3.3, enjoys homogeneity, sub-additivity, and co-monotonic additivity. Its associated function is continuously differentiable under moderate assumptions on the joint distribution of the variables X_i (cf. Tasche, 1999, 2002).*

Assume that ρ is a risk measure that has these four properties. Then

(i) by sub-additivity, $\mathrm{DI}_\rho(X) \le 1$

(ii) by co-monotonic additivity, $\mathrm{DI}_\rho(X) \approx 1$ *indicates that $X_1, \ldots, X_n$ are “almost” co-monotonic (i.e. strongly dependent)*

(iii) by (2.9b), $\mathrm{DI}_\rho(X_i \,|\, X) \le 1$

[3] For standard deviation based risk measures (2.13b) obtains if and only if V and W are fully linearly correlated (i.e. correlated with correlation coefficient 1). Full linear correlation implies co-monotonicity but co-monotonicity does not imply full correlation. Standard deviation based worst-case scenarios, therefore, might be considered “non-representative”.

(iv) *by Proposition 2.1,* $\mathrm{DI}_\rho(X_i \,|\, X) < \mathrm{DI}_\rho(X)$ *implies that there is some* $\epsilon_i > 0$ *such that* $\mathrm{DI}_\rho(X + h\,X_i) < \mathrm{DI}_\rho(X)$ *for* $0 < h < \epsilon_i$.

With regard to conclusion (ii) in Remark 2.2, a portfolio with a diversification index close to 100% might be considered to have high risk concentration whereas a portfolio with a low diversification index might be considered well diversified. However, it is not easy to specify how far from 100% a diversification index should be for implying that the portfolio is well diversified.

Conclusion (iv) in Remark 2.2 could be more useful for such a distinction between concentrated and diversified portfolios, because marginal diversification indices indicate rather diversification potential than "absolute" diversification. In this sense, a portfolio with high unrealized diversification potential could be regarded as concentrated.

3 Practice

On principle, formula (2.6) can be applied directly for calculating Euler risk contributions. It has turned out, however, that for some popular families of risk measures it is possible to derive closed-form expressions for the involved derivative. In this section, results are presented for standard deviation based risk measures (Section 3.1), Value-at-Risk (Section 3.2), and Expected Shortfall (Section 3.3)[4]. In Section 3.4, we discuss in some detail how to implement (2.6) in the case of VaR when the underlying distribution has to be inferred from a sample. In the following, the notation of Section 2.1 is adopted.

3.1 Standard deviation based risk measures

We consider here the family of risk measures σ_c, $c > 0$ given by

$$\sigma_c(X) \;=\; c\,\sqrt{\mathrm{var}[X]} \;=\; c\,\sqrt{\mathrm{E}[(X - \mathrm{E}[X])^2]}. \tag{3.1}$$

It is common to choose the constant c in such a way that

$$\mathrm{P}[X \le \mathrm{E}[X] - \sigma_c(X)] \;\le\; 1 - \alpha, \tag{3.2}$$

where α denotes some – usually large – probability (like 99% or 99.95%). Sometimes this is done assuming that X is normally distributed. A robust alternative would be an application of the one-tailed Chebychev-inequality:

$$\mathrm{P}[X \le \mathrm{E}[X] - \sigma_c(X)] \;\le\; \frac{1}{1 + c^2}. \tag{3.3}$$

Solving $1/(1 + c^2) = 1 - \alpha$ for c will then ensure that (3.2) obtains for all X with finite variance. Note that, however, this method for determining c will yield much higher values of c than the method based on a normal-distribution assumption.

The risk measures σ_c, $c > 0$ are homogeneous of degree 1 and sub-additive, but not co-monotonic additive. For $X = \sum_{i=1}^n X_i$ the Euler contributions according to Remark 2.1 can readily be calculated by differentiation:

$$\sigma_c(X_i \,|\, X) \;=\; c\,\frac{\mathrm{cov}[X_i,\, X]}{\sqrt{\mathrm{var}[X]}}. \tag{3.4a}$$

In case that X_i is given as $g_i - L_i$ (cf. (2.3)), we have $\sigma_c(X) = \sigma_c(L)$ and

$$\sigma_c(X_i \,|\, X) \;=\; c\,\frac{\mathrm{cov}[L_i,\, L]}{\sqrt{\mathrm{var}[L]}}. \tag{3.4b}$$

[4] With respect to other classes of risk measures see, e.g., Fischer (2003) for a discussion of derivatives of one-sided moment measures and Tasche (2002) for a suggestion of how to apply (3.7a) to spectral risk measures.

3.2 Value-at-Risk

For any real-valued random variable Y and $\gamma \in (0,1)$ define the γ-quantile of Y by

$$q_\gamma(Y) \;=\; \min\{y : \mathrm{P}[Y \le y] \ge \gamma\}. \tag{3.5a}$$

If Y has a strictly increasing and continuous distribution function $F(y) = \mathrm{P}[Y \le y]$, quantiles of Y can be expressed by the inverse function of F:

$$q_\gamma(Y) \;=\; F^{-1}(\gamma). \tag{3.5b}$$

For a portfolio-wide profit/loss variable $X = \sum_{i=1}^n X_i$ the *Value-at-Risk (VaR)* of X at confidence level α (α usually close to 1) is defined as the α-quantile of $-X$:

$$\mathrm{VaR}_\alpha(X) \;=\; q_\alpha(-X). \tag{3.6}$$

VaR as a risk measure is homogeneous of degree 1 and co-monotonic additive but not in general sub-additive. Under some smoothness conditions (see Gouriéroux et al., 2000, or Tasche, 1999, Section 5.2), a general formula can be derived for the Euler contributions to $\mathrm{VaR}_\alpha(X)$ according to Remark 2.1. These smoothness conditions, in particular, imply that X has a density. The formula for the Euler VaR-contributions reads

$$\mathrm{VaR}_\alpha(X_i \,|\, X) \;=\; -\mathrm{E}[X_i \,|\, X = -\mathrm{VaR}_\alpha(X)], \tag{3.7a}$$

where $\mathrm{E}[X_i \,|\, X]$ denotes the *conditional expectation* of X_i given X. In case that X_i is given as $g_i - L_i$ (cf. (2.3)), we have $\mathrm{VaR}_\alpha(X) = q_\alpha(L) - \sum_{i=1}^n g_i$ and

$$\mathrm{VaR}_\alpha(X_i \,|\, X) \;=\; \mathrm{E}[L_i \,|\, L = q_\alpha(L)] - g_i. \tag{3.7b}$$

Often, it is not VaR itself that is of interest but rather *Unexpected Loss:*

$$\mathrm{UL}_{\mathrm{VaR},\alpha}(X) \;=\; \mathrm{VaR}_\alpha(X - \mathrm{E}[X]) \;=\; \mathrm{VaR}_\alpha(X) + \mathrm{E}[X]. \tag{3.8a}$$

In terms of $X_i = g_i - L_i$, Equation (3.8a) reads

$$\mathrm{UL}_{\mathrm{VaR},\alpha}(X) \;=\; \mathrm{VaR}_\alpha(\mathrm{E}[L] - L) \;=\; q_\alpha(L) - \mathrm{E}[L]. \tag{3.8b}$$

Having in mind that the Euler contribution of X_i to $\mathrm{E}[X]$ is obviously $\mathrm{E}[X_i]$, the formulae for the Euler contributions to $\mathrm{UL}_{\mathrm{VaR},\alpha}(X)$ are obvious from Equations (3.7a), (3.7b), (3.8a), and (3.8b).

In general, the conditional expectation of X_i given X cannot easily be calculated or estimated. For some exceptions from this observation see Tasche (2004a) or Tasche (2006). As the conditional expectation of X_i given X can be interpreted as the *best prediction* of X_i by X in a least squares context, approximation of $\mathrm{VaR}_\alpha(X_i \,|\, X)$ by *best linear predictions* of X_i by X has been proposed. Linear approximation of the right-hand side of (3.7a) by X and a constant yields

$$\mathrm{VaR}_\alpha(X_i \,|\, X) \;\approx\; \frac{\mathrm{cov}[X_i,\, X]}{\mathrm{var}[X]}\, \mathrm{UL}_{\mathrm{VaR},\alpha}(X) - \mathrm{E}[X_i]. \tag{3.9}$$

The approximation in (3.9) can be improved by additionally admitting quadratic or other non-linear transformations of X as regressors (cf. Tasche and Tibiletti, 2004, Section 5). All such regression-based approximate Euler contributions to VaR satisfy the full allocation property of Definition 2.2 but are not RORAC compatible. See Section 3.4 for an approach to the estimation of (3.7a) that yields RORAC compatibility.

3.3 Expected shortfall

For a portfolio-wide profit/loss variable $X = \sum_{i=1}^n X_i$ the *Expected Shortfall (ES)*[5] of X at confidence level α (α usually close to 1) is defined as an average of VaRs of X at level α and higher:

$$\mathrm{ES}_\alpha(X) \;=\; \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(X)\, du. \tag{3.10a}$$

[5] The denotation "Expected Shortfall" was proposed by Acerbi and Tasche (2002). A common alternative denotation is "Conditional Value-at-Risk (CVaR)" that was suggested by Rockafellar and Uryasev (2002).

ES as a risk measure is homogeneous of degree 1, co-monotonic additive and sub-additive. Under some smoothness conditions (see Tasche, 1999, Section.5.3), a general formula can be derived for the Euler contributions to $\mathrm{ES}_\alpha(X)$ according to Remark 2.1. These smoothness conditions, in particular, imply that X has a density. In that case, ES can equivalently be written as

$$\mathrm{ES}_\alpha(X) \;=\; -\mathrm{E}[X \,|\, X \le -\mathrm{VaR}_\alpha(X)]. \tag{3.10b}$$

The formula for the Euler ES-contributions reads

$$\begin{aligned}\mathrm{ES}_\alpha(X_i \,|\, X) \;&=\; -\mathrm{E}[X_i \,|\, X \le -\mathrm{VaR}_\alpha(X)]\\ &=\; -(1-\alpha)^{-1}\mathrm{E}[X_i\, \mathbf{1}_{\{X \le -\mathrm{VaR}_\alpha(X)\}}].\end{aligned} \tag{3.11a}$$

Note that $\mathrm{E}[X_i \,|\, X \le -\mathrm{VaR}_\alpha(X)]$, in contrast to $\mathrm{E}[X_i \,|\, X]$ from (3.7a), is an *elementary conditional expectation* because the conditioning event has got a positive probability to occur. In case that X_i is given as $g_i - L_i$ (cf. (2.3)), we have $\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 q_u(L)\,du - \sum_{i=1}^n g_i$ and

$$\mathrm{ES}_\alpha(X_i \,|\, X) \;=\; \mathrm{E}[L_i \,|\, L \ge q_\alpha(L)] - g_i. \tag{3.11b}$$

Often, it is not ES itself that is of interest but rather *Unexpected Loss:*

$$\mathrm{UL}_{\mathrm{ES},\alpha}(X) \;=\; \mathrm{ES}_\alpha(X - \mathrm{E}[X]) \;=\; \mathrm{ES}_\alpha(X) + \mathrm{E}[X]. \tag{3.12a}$$

In terms of $X_i = g_i - L_i$, Equation (3.12a) reads

$$\mathrm{UL}_{\mathrm{ES},\alpha}(X) \;=\; \mathrm{ES}_\alpha(\mathrm{E}[L] - L) \;=\; \frac{1}{1-\alpha}\int_\alpha^1 q_u(L)\,du - \mathrm{E}[L]. \tag{3.12b}$$

Having in mind that the Euler contribution of X_i to $\mathrm{E}[X]$ is obviously $\mathrm{E}[X_i]$, the formulae for the Euler contributions to $\mathrm{UL}_{\mathrm{ES},\alpha}(X)$ are obvious from Equations (3.11a), (3.11b), (3.12a), and (3.12b).

In contrast to Euler contributions to VaR, thanks to representation (3.11a), estimation of Euler contributions to ES is quite straightforward.

3.4 Risk measures for sample data

In most circumstances, portfolio loss distributions cannot be calculated analytically but have to be estimated from simulated or historical sample data. For a portfolio of n assets, as specified in Section 2.1, the sample data might be given as n-dimensional points

$$(x_{1,1}, \ldots, x_{n,1}), \ldots, (x_{1,N}, \ldots, x_{n,N}), \tag{3.13}$$

where $x_{i,k}$ denotes the profit/loss of asset i in the k-th observation (of N). Each data point $(x_{1,k}, \ldots, x_{n,k})$ would be interpreted as a realisation of the profit/loss random vector $(X_1, \ldots, X_n)$. The portfolio-wide profit/loss in the k-th observation would then be obtained as $x_k = \sum_{i=1}^n x_{i,k}$. If the sample is large and the observations can be assumed independent, by the law of large numbers the empirical measure

$$\begin{aligned}\widehat{\mathrm{P}}_N(A) \;&=\; \frac{1}{N}\sum_{k=1}^N \delta_A(x_{1,k}, \ldots, x_{n,k}), \quad A \subset \mathbb{R}^n \text{ measurable},\\ \delta_A(x_{1,k}, \ldots, x_{n,k}) \;&=\; \begin{cases} 1, & \text{if } (x_{1,k}, \ldots, x_{n,k}) \in A;\\ 0, & \text{otherwise}\end{cases}\end{aligned} \tag{3.14}$$

will approximate the joint distribution $\mathrm{P}[(X_1, \ldots, X_n) \in A]$ of the assets' profits/losses X_i.

Denote by $(\widehat{X}_1, \ldots, \widehat{X}_n)$ the profit/loss random vector under the empirical measure $\widehat{\mathrm{P}}_N$. Let $\widehat{X} = \sum_{i=1}^n \widehat{X}_i$ be the portfolio-wide profit/loss under the empirical measure $\widehat{\mathrm{P}}_N$. In the cases of standard deviation based risk measures and Expected Shortfall, then *statistically consistent* estimators for the risk contributions according to (3.4a) and (3.11a) respectively can be obtained by simply substituting $\widehat{X}, \widehat{X}_i$ for X, X_i. This naive approach does not work for Euler contributions to VaR according to (3.7a), if X has a continuous distribution. In this case, smoothing of the empirical measure (*kernel estimation*) as described in the following can help.

Let ξ be a random variable which is independent of $(\widehat{X}_1, \ldots, \widehat{X}_n)$ and has a continuous density (or *kernel*) φ (ξ standard normal would be a good and convenient choice). Fix some $b > 0$ (the *bandwidth*). Then $\widehat{X} + b\,\xi$ has the density

$$\widehat{f}_b(x) \;=\; \widehat{f}_{b,x_1,\ldots,x_N}(x) \;=\; \frac{1}{b\,N}\sum_{k=1}^{N}\varphi\big(\tfrac{x-x_k}{b}\big). \tag{3.15}$$

Actually, (3.15) represents the well-known *Rosenblatt-Parzen* estimator of the density of X. If the kernel φ and the density of X are appropriately "smooth" (see, e.g., Pagan and Ullah, 1999, Theorem 2.5 for details), it can be shown for $b_N \to 0$, $b_N\,N \to \infty$ that $\widehat{f}_{b_N}(x)$ is a pointwise mean-squared consistent estimator of the density of X. Whereas the choice of the kernel φ, subject to some minimum conditions, is not too important for the efficiency of the Rosenblatt-Parzen estimator, the appropriate choice of the bandwidth is crucial. *Silverman*'s rule of thumb (cf. (2.52) in Pagan and Ullah, 1999)

$$b \;=\; 0.9\,\min(\sigma, R/1.34)\,N^{-1/5} \tag{3.16}$$

is known to work quite well in many circumstances. In (3.16), σ and R denote the standard deviation and the interquartile range respectively of the sample $x_1, \ldots, x_N$. In the case where X is not heavy-tailed (as is the case for credit portfolio loss distributions), another quite promising and easy-to-implement method for the bandwidth selection is *Pseudo Likelihood Cross Validation*. See, for instance, Turlach (1993) for an overview of this and other selection methods.

In the context of the Euler allocation, the big advantage with the kernel estimation approach is that the smoothness conditions from Tasche (1999, Section 5.2) are obtained for $\xi, \widehat{X}_1, \ldots, \widehat{X}_n$. As a consequence, if follows from Tasche (1999, Lemma 5.3) that

$$\begin{aligned}
\mathrm{VaR}_\alpha(X_i\,|\,X) \;&\approx\; \mathrm{VaR}_\alpha(\widehat{X}_i\,|\,\widehat{X} + b\,\xi)\\
&=\; \frac{d\,\mathrm{VaR}_\alpha}{d\,h}(\widehat{X} + b\,\xi + h\,\widehat{X}_i)\big|_{h=0}\\
&=\; -\,\mathrm{E}[\widehat{X}_i\,|\,\widehat{X} + b\,\xi = -\mathrm{VaR}_\alpha(\widehat{X} + b\,\xi)]\\
&=\; -\,\frac{\sum_{k=1}^{N} x_{i,k}\,\varphi\big(\frac{-\mathrm{VaR}_\alpha(\widehat{X}+b\,\xi)-x_k}{b}\big)}{\sum_{k=1}^{N}\varphi\big(\frac{-\mathrm{VaR}_\alpha(\widehat{X}+b\,\xi)-x_k}{b}\big)}.
\end{aligned} \tag{3.17}$$

The right-hand side of (3.17) is just the *Nadaraya-Watson* kernel estimator of $\mathrm{E}[X_i\,|\,X = -\mathrm{VaR}_\alpha(X)]$. Note that it is clear from the derivation of (3.17) in the context of the empirical measure (3.14) that the bandwidth b in (3.17) should be the same as in (3.15). By construction, we obtain for the sum of the approximate Euler contributions to VaR according to (3.17)

$$\begin{aligned}
\sum_{i=1}^{n}\mathrm{VaR}_\alpha(\widehat{X}_i\,|\,\widehat{X} + b\,\xi) \;&=\; \mathrm{VaR}_\alpha(\widehat{X} + b\,\xi) - \mathrm{VaR}_\alpha(\xi\,|\,\widehat{X} + b\,\xi)\\
&=\; \mathrm{VaR}_\alpha(\widehat{X} + b\,\xi) + \mathrm{E}[\xi\,|\,\widehat{X} + b\,\xi = -\mathrm{VaR}_\alpha(\widehat{X} + b\,\xi)].
\end{aligned} \tag{3.18}$$

The sum of the approximate Euler contributions therefore differs from natural estimates of $\mathrm{VaR}\alpha(X)$ such as $\mathrm{VaR}_\alpha(\widehat{X})$ or $\mathrm{VaR}_\alpha(\widehat{X} + b\,\xi)$. Practical experience shows that the difference tends to be small. Some authors (e.g. Epperlein and Smillie, 2006) suggest to account for this difference by an appropriate multiplier. Another way to deal with the issue could be to take the left-hand side of (3.18) as an estimate for $\mathrm{VaR}_\alpha(X)$.

Yamai and Yoshiba (2001) found that estimates for ES and VaR Euler contributions are very volatile. See Glasserman (2005) and Merino and Nyfeler (2004) for methods to tackle this problem for ES by importance sampling and Tasche (2007) for an approach to Euler VaR contribution estimation by importance sampling.

4 Conclusions

Among the many methodologies that financial institutions apply for their internal capital allocation process the so-called *Euler allocation* is especially appealing for its economic foundation. There are a lot of papers that provide partial information on the Euler allocation but a comprehensive overview seems to be missing so far. In this paper, we have tried to fill this gap to some extent. In particular, besides emphasizing the economic meaning of Euler allocation, we have discussed its potential application to the detection of risk concentrations. We have moreover demonstrated that there is a natural relationship between Euler contributions to VaR and kernel estimators for conditional expectations.

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A Euler's theorem on homogeneous functions

In this paper, the focus is on *homogeneous* risk measures and functions.

Definition A.1 *A risk measure* ρ *in the sense of* (2.2) *is called* homogeneous of degree τ *if for any* $h > 0$ *the following equation obtains:*

$$\rho(h\,X) \;=\; h^\tau\,\rho(X).$$

A function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ *is called* homogeneous of degree τ *if for any* $h > 0$ *and* $u \in U$ *with* $h\,u \in U$ *the following equation holds:*

$$f(h\,u) \;=\; h^\tau\,f(u).$$

Note the function f_ρ corresponding by (2.5) to the risk measure ρ is homogeneous of degree τ is ρ is homogeneous of degree τ.

In the case of continuously differentiable functions, homogeneous functions can be described by Euler's theorem.

Theorem A.1 (Euler's theorem on homogeneous functions) *Let* $U \subset \mathbb{R}^n$ *be an open set and* $f : U \to \mathbb{R}$ *be a continuously differentiable function. Then* f *is homogeneous of degree* τ *if and only if it satisfies the following equation:*

$$\tau\,f(u) \;= \sum_{i=1}^n u_i\,\frac{\partial\,f(u)}{\partial\,u_i}, \qquad u = (u_1, \ldots, u_n) \in U,\; h > 0.$$

Remark A.1 *It is easy to show that* $\frac{\partial\,f}{\partial\,u_i}$ *is homogeneous of degree* $\tau - 1$ *if* f *is homogeneous of degree* τ*. From this observation follows that if* f *is homogeneous of degree 1 and continuously differentiable for* $u = 0$ *then* f *is a linear function (i.e. with constant partial derivatives). Often, therefore, the homogeneous functions relevant for risk management are not differentiable in* $u = 0$*.*

Functions $f : U \subset \mathbb{R}^n \to \mathbb{R}$ that are homogeneous of degree 1 are *convex*, i.e.

$$f(\eta\,u + (1-\eta)\,v) \;\le\; \eta\,f(u) + (1-\eta)\,f(v), \qquad u, v \in U,\; \eta \in [0,1], \tag{A.1a}$$

if and only if they are *sub-additive*, i.e.

$$f(u+v) \;\le\; f(u) + f(v), \qquad u, v \in U. \tag{A.1b}$$

Theorem A.1 implies a useful characterisation of sub-additivity for continuously differentiable functions that are homogeneous of degree 1.

Corollary A.1 *Let* $U \subset \mathbb{R}^n$ *be an open set and* $f : U \to \mathbb{R}$ *be a continuously differentiable function that is homogeneous of degree 1. Then* f *is sub-additive if and only if the following inequality holds:*

$$\sum_{i=1}^n u_i\,\frac{\partial\,f(u+v)}{\partial\,u_i} \;\le\; f(u), \qquad u, u+v \in U.$$

See Proposition 2.5 of Tasche (2002) for a proof of Corollary A.1.