Granularity Adjustment for Risk Measures:

Systematic vs Unsystematic Risks

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Abstract

The granularity principle [Gordy (2003)] allows for closed form expressions of the risk measures of a large portfolio at order $1/n$, where n is the portfolio size. The granularity principle yields a decomposition of such risk measures that highlights the different effects of systematic and unsystematic risks. This paper derives the granularity adjustment of the Value-at-Risk (VaR), the Expected Shortfall and the other distortion risk measures for both static and dynamic risk factor models. The systematic factor can be multidimensional. The methodology is illustrated by several examples, such as the stochastic drift and volatility model, or the dynamic factor model for joint analysis of default and loss given default.

Keywords: Value-at-Risk, Granularity, Large Portfolio, Credit Risk, Systematic Risk, Loss Given Default, Basel 2 Regulation, Credibility Theory.

JEL classification: G12, C23.

1 Introduction

Risk measures such as the Value-at-Risk (VaR), the Expected Shortfall (also called Tail VaR) and more generally the Distortion Risk Measures (DRM) [Wang (2000)] are the basis of the new risk management policies and regulations in both Finance (Basel 2) and Insurance (Solvency 2). These measures are used to define the minimum capital required to hedge risky investments (Pillar 1 in Basel 2). They are also used to monitor the risk by means of internal risk models (Pillar 2 in Basel 2).

These risk measures have in particular to be computed for large portfolios of individual contracts, which can be loans, mortgages, life insurance contracts, or Credit Default Swaps (CDS), and for derivative assets written on such large portfolios, such as Mortgage Backed Securities, Collateralized Debt Obligations, derivatives on a CDS index (such as iTraxx, or CDX), Insurance Linked Securities, or longevity bonds. The value of a portfolio risk measure is often difficult to derive even numerically due to

i) the large size of the support portfolio, which can include from about one hundred [1] to several thousands of individual contracts;

ii) the nonlinearity of individual risks, such as default, recovery, claim occurrence, prepayment, surrender, lapse;

iii) the need to take into account the dependence between individual risks induced by the systematic components of these risks.

The granularity principle has been introduced for static single factor models during the discussion on the New Basel Capital Accord [BCBS (2001)], following the contributions by Gordy (2003) and Wilde (2001). The granularity principle allows for closed form expressions of the risk measures for large portfolios at order $1/n$, where n denotes the portfolio size. More precisely, any portfolio risk measure can be decomposed as the sum of an asymptotic risk measure corresponding to an infinite portfolio size and $1/n$ times an adjustment term. The asymptotic portfolio risk measure, called Cross-Sectional Asymptotic (CSA) risk measure, captures the non diversifiable effect of systematic risks on the portfolio. The adjustment term, called Granularity Adjustment (GA), summarizes the effect of the individual specific risks and their cross-effect with systematic risks,

[1] This corresponds to the number of names included in the iTraxx, or CDX indexes.

when the portfolio size is large, but finite.

Despite its analytical tractability and intuitive appeal, the static single risk factor model is too restrictive to capture the complexity and dynamics of systematic risks. Multiple dynamic factors are needed for a joint analysis of stochastic drift and volatility, of default and loss given default, to model country and industrial sector specific effects, to monitor the risk of loans with guarantees, when the guarantors themselves can default [Ebert, Lutkebohmert (2009)], or to distinguish between trend and cycle effects. Motivated by these applications, the purpose of our paper is to extend the granularity approach to a dynamic multiple factor framework.

The static risk factor model is introduced in Section 2, and the granularity adjustment of the VaR is given in Section 3. The GA for the VaR can be used to derive easily the GA for any other Distortion Risk Measure, including for instance Expected Shortfall. Section 4 provides the granularity adjustment for a variety of static single and multiple risk factor models. The analysis is extended to dynamic risk factor models in Section 5. In the dynamic framework, two granularity adjustments are required. The first GA concerns the conditional VaR with current factor value assumed to be observed. The second GA takes into account the unobservability of the current factor value. This new decomposition relies on recent results on the granularity principle applied to nonlinearing filtering problems [Gagliardini, Gouriéroux (2010)a]. Whereas the initial version of the Basel 2 regulation has focused on modeling the stochastic probability of default assuming a deterministic loss given default, the most advanced approaches have to account for the uncertainty in the recovery rate and its correlation with the probability of default. In Section 6 we introduce a dynamic two-factor model with stochastic probability of default and loss given default, and derive the patterns of the granularity adjustment. Section 7 concludes. The theoretical derivations of the granularity adjustments are done in the Appendices.

2 Static Risk Factor Model

We first consider the static risk factor model to focus on the individual risks and their dependence structure. We omit the unnecessary time index.

2.1 Homogenous Portfolio

Let us assume that the individual risks (e.g. asset values, or default indicators) depend on some common factors and on individual specific effects:

$$y_i = c(F, u_i), \quad i = 1, \ldots, n, \tag{2.1}$$

where y_i denotes the individual risk, F the systematic factor and u_i the idiosyncratic term. Both F and u_i can be multidimensional, whereas y_i is one-dimensional. Variables F and u_i satisfy the following assumptions:

Distributional Assumptions: *For any portfolio size n:*

A.1: *F and $(u_1, \ldots, u_n)$ are independent.*

A.2: *$u_1, \ldots, u_n$ are independent and identically distributed.*

The portfolio of individual risks is homogenous, since the joint distribution of $(y_1, \ldots, y_n)$ is invariant by permutation of the n individuals, for any n. This exchangeability property of the individual risks is equivalent to the fact that variables $y_1, \ldots, y_n$ are independent, identically distributed conditional on some factor F [de Finetti (1931), Hewitt, Savage (1955)]. When the unobservable systematic factor F is integrated out, the individual risks become dependent.

2.2 Examples

We describe below simple examples of static Risk Factor Model (RFM) (see Section 4 for further examples).

Example 2.1: Linear Single-Factor Model

We have:

$$y_i = F + u_i,$$

where the specific error terms u_i are Gaussian $N(0, \sigma^2)$ and the factor F is Gaussian $N(\mu, \eta^2)$. Since $Corr\ (y_i, y_j) = \eta^2/(\eta^2+\sigma^2)$, for $i \neq j$, the common factor creates the (positive) dependence between individual risks, whenever $\eta^2 \neq 0$. This model has been used rather early in the literature on individual risks. For instance, it is the Buhlmann model considered in actuarial science and is the basis for credibility theory [Buhlmann (1967), Buhlmann, Straub (1970)].

Example 2.2: The Single Risk Factor Model for Default

The individual risk is the default indicator, that is $y_i = 1$, if there is a default of individual i, and $y_i = 0$, otherwise. This risk variable is given by:

$$y_i = \begin{cases} 1, & \text{if } F + u_i < 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $F \sim N(\mu, \eta^2)$ and $u_i \sim N(0, \sigma^2)$. The quantity $F + u_i$ is often interpreted as a log asset/liability ratio, when i is a company [see e.g. Merton (1974), Vasicek (1991)]. Thus, the company defaults when the asset value becomes smaller than the amount of debt.

The basic specifications in Examples 2.1 and 2.2 can be extended by introducing additional individual heterogeneity, or multiple factors.

Example 2.3: Model with Stochastic Drift and Volatility

The individual risks are such that:

$$y_i = F_1 + (\exp F_2)^{1/2} u_i,$$

where F_1 (resp. F_2) is a common stochastic drift factor (resp. stochastic volatility factor). When y_i is an asset return, we expect factors F_1 and F_2 to be dependent, since the (conditional) expected return generally contains a risk premium.

Example 2.4: Linear Single Risk Factor Model with Beta Heterogeneity

This is a linear factor model, in which the individual risks may have different sensitivities (called betas) to the systematic factor. The model is:

$$y_i = \beta_i F + v_i,$$

where $u_i = (\beta_i, v_i)'$ is bidimensional. In particular, the betas are assumed unobservable and are included among the idiosyncratic risks. This type of model is the basis of Arbitrage Pricing Theory (APT) [see e.g. Ross (1976), and Chamberlain, Rothschild (1983), in which similar assumptions are introduced on the beta coefficients].

3 Granularity Adjustment for Portfolio Risk Measures

3.1 Portfolio Risk

Let us consider an homogenous portfolio including n individual risks. The total portfolio risk is:

$$W_n = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} c(F, u_i). \tag{3.1}$$

The total portfolio risk can correspond to a profit and loss (P&L), for instance when y_i is an asset return and W_n/n the equally weighted portfolio return. In other cases, it corresponds to a loss and profit (L&P), for instance when y_i is a default indicator and W_n/n the portfolio default frequency [2]. As usual, we pass from a P&L to a L&P by a change of sign [3]. The quantile of W_n at a given risk level is used to define a VaR (resp. the opposite of a VaR), if W_n is a L&P (resp. a P&L).

The distribution of W_n is generally unknown in closed form due to the risks dependence and the aggregation step. The density of W_n involves integrals with a large dimension, which can reach $dim(F) + n\ dim(u) - 1$. Therefore, the quantiles of the distribution of W_n, can also be difficult to compute [4]. To address this issue we consider a large portfolio perspective.

3.2 Asymptotic Portfolio Risk

The standard limit theorems such as the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT) cannot be applied directly to the sequence $y_1, \ldots, y_n$ due to the common factors. However, LLN and CLT can be applied conditionally on the factor values, under Assumptions A.1 and A.2. This is the condition of infinitely fine grained portfolio [5] in the Basel 2 terminology

[2]The results of the paper are easily extended to obligors with different exposures A_i, say. In this case we have $W_n = \sum_{i=1}^{n} A_i y_i = \sum_{i=1}^{n} A_i c(F, u_i) = \sum_{i=1}^{n} c^*(F, u_i^*)$, where the idiosyncratic risks $u_i^* = (u_i, A_i)$ contain the individual shocks u_i and the individual exposures A_i [see e.g. Emmer, Tasche (2005) in a particular case].

[3]This means that asset returns will be replaced by opposite asset returns, that are returns for investors with short positions.

[4]The VaR can often be approximated by simulations [see e.g. Glasserman, Li (2005)], but these simulations are very time consuming, if the portfolio size is large and the risk level of the VaR is small, especially in dynamic factor models.

[5]Loosely speaking, "the portfolio is infinitely fine grained, when the largest individual exposure accounts for an infinitely small share of the total portfolio exposure" [Ebert, Lutkebohmert (2009)]. This condition is satisfied under Assumptions A.1 and A.2.

[BCBS (2001)].

Let us denote by

$$m(F) = E[y_i|F] = E[c(F, u_i)|F], \tag{3.2}$$

the conditional individual expected risk, and

$$\sigma^2(F) = V[y_i|F] = V[c(F, u_i)|F], \tag{3.3}$$

the conditional individual volatility. By applying the CLT conditional on F, we get:

$$W_n/n = m(F) + \sigma(F)\frac{X}{\sqrt{n}} + O(1/n), \tag{3.4}$$

where X is a standard Gaussian variable independent of factor F. The term at order $O(1/n)$ is zero mean, conditional on F, since W_n/n is a conditionally unbiased estimator of $m(F)$.

Expansion (3.4) differs from the expansion associated with the standard CLT. Whereas the first term of the expansion is constant, equal to the unconditional mean in the standard CLT, it is stochastic in expansion (3.4) and linked with the second term of the expansion by means of factor F. Moreover, each term in the expansion depends on the factor value, but also on the distribution of idiosyncratic risk by means of functions $m(.)$ and $\sigma(.)$. By considering expansion (3.4), the initial model with $dim(F) + n\ dim(u)$ dimensions of uncertainty is approximated by a 3-dimensional model, with uncertainty summarized by means of $m(F)$, $\sigma(F)$ and X.

3.3 Granularity principle

The granularity principle has been introduced for static single risk factor models by Gordy in 2000 for application in Basel 2 [Gordy (2003)]. We extend below this principle to multiple factor models. The granularity principle requires several steps, which are presented below for a loss and profit variable.

i) A standardized risk measure

Instead of the VaR of the portfolio risk, which explodes with the portfolio size, it is preferable to consider the VaR per individual risk (asset) included in this portfolio. Since by Assumptions A.1-A.2 the individuals are exchangeable and a quantile function is homothetic, the VaR per individual risk is simply a quantile of W_n/n. The VaR at risk level $\alpha^* = 1 - \alpha$ is denoted by $VaR_n(\alpha)$ and is defined by the condition:

$$P[W_n/n < VaR_n(\alpha)] = \alpha, \tag{3.5}$$

where α is a positive number close to 1, typically $\alpha = 95\%$, 99%, 99.5%, which correspond to probabilities of large losses equal to $\alpha^* = 5\%$, 1%, 0.5%, respectively.

ii) The CSA risk measure

Vasicek [Vasicek (1991)] proposed to first consider the limiting case of a portfolio with infinite size. Since

$$\lim_{n\to\infty} W_n/n = m(F), \quad a.s., \tag{3.6}$$

the infinite size portfolio is not riskfree. Indeed, **the systematic risk is undiversifiable**. We deduce that the CSA risk measure:

$$VaR_\infty(\alpha) = \lim_{n\to\infty} VaR_n(\alpha), \tag{3.7}$$

is the α-quantile associated with the systematic component of the portfolio risk:

$$P[m(F) < VaR_\infty(\alpha)] = \alpha. \tag{3.8}$$

The CSA risk measure is suggested in the Internal Ratings Based (IRB) approach of Basel 2 for minimum capital requirement. This approach neglects the effect of unsystematic risks in a portfolio of finite size.

iii) Granularity Adjustment for the risk measure

The main result in granularity theory applied to risk measures provides the next term in the asymptotic expansion of $VaR_n(\alpha)$ with respect to n, for large n. It is given below for a multiple factor model.

Proposition 1: *In a static RFM, we have:*

$$VaR_n(\alpha) = VaR_\infty(\alpha) + \frac{1}{n} GA(\alpha) + o(1/n),$$

where:

$$\begin{aligned} GA(\alpha) &= -\frac{1}{2}\left\{\frac{d\log g_\infty}{dw}[VaR_\infty(\alpha)]E[\sigma^2(F)|m(F)=VaR_\infty(\alpha)] \right. \\ &\qquad \left. + \frac{d}{dw}E[\sigma^2(F)|m(F)=w]\Big|_{w=VaR_\infty(\alpha)}\right\} \\ &= -\frac{1}{2}E\left[\sigma^2(F)|m(F)=VaR_\infty(\alpha)\right]\left\{\frac{d\log g_\infty}{dw}[VaR_\infty(\alpha)] \right. \\ &\qquad \left. + \frac{d}{dw}\log E\left[\sigma^2(F)|m(F)=w\right]\Big|_{w=VaR_\infty(\alpha)}\right\}, \end{aligned}$$

and g_∞ [resp. $VaR_\infty(.)$] denotes the probability density function (resp. the quantile function) of the random variable $m(F)$.

Proof: See Appendix 1.

The GA in Proposition 1 depends on the tail magnitude of the systematic risk component $m(F)$ by means of $\frac{d\log g_\infty}{dw}[VaR_\infty(\alpha)]$, which is expected to be negative. The GA depends also on $\frac{d}{dw}\log E\left[\sigma^2(F)|m(F)=w\right]\Big|_{w=VaR_\infty(\alpha)}$, which is a measure of the reaction of the individual volatility to shocks on the individual drift. When $y_{i,t}$ is the opposite of an asset return, this reaction function is expected to be nonlinear and increasing for positive values of $m(F)$, according to the leverage effect interpretation [Black (1976)].

When the tail effect is larger than the leverage effect, the GA is positive, which implies an increase of the required capital compared to the CSA risk measure. In the special case of independent stochastic drift and volatility, the GA reduces to $GA(\alpha) = -\frac{1}{2}E\left[\sigma^2(F)|m(F)=VaR_\infty(\alpha)\right]$ $\frac{d\log g_\infty}{dw}[VaR_\infty(\alpha)]$, which is generally positive. The adjustment involves both the tail of the systematic risk component and the expected conditional variability of the individual risks.

The asymptotic expansion of the VaR in Proposition 1 is important for several reasons. i) The computation of quantities $VaR_\infty(\alpha)$ and $GA(\alpha)$ does not require the evaluation of large dimensional integrals. Indeed, $VaR_\infty(\alpha)$ and $GA(\alpha)$ involve the distribution of transformations $m(F)$ and $\sigma^2(F)$ of the systematic factor only, which are independent of the portfolio size n.

ii) The second term in the expansion is of order $1/n$, and not $1/\sqrt{n}$ as might have been expected from the Central Limit Theorem. This implies that the approximation $VaR_\infty(\alpha) + \frac{1}{n}GA(\alpha)$ is likely rather accurate, even for rather small values of n such as $n = 100$.

iii) The expansion is valid for single as well as multiple factor models.

iv) The expansion is easily extended to the other Distortion Risk Measures, which are weighted averages of VaR:

$$DRM_n(H) = \int VaR_n(u)dH(u), \text{ say,}$$

where H denotes the distortion measure [Wang (2000)]. The granularity adjustment for the DRM is simply:

$$\frac{1}{n}\int GA(u)dH(u).$$

In particular, the Expected Shortfall at confidence level α corresponds to the distortion measure with cumulative distribution function $H(u;\alpha) = (u-\alpha)^+/(1-\alpha)$, and the granularity adjustment is

$$\frac{1}{n(1-\alpha)}\int_\alpha^1 GA(u)du,$$

that is an average of the granularity adjustments for VaR above level α.

v) Proposition 1 can be used to investigate the difficult task of aggregation of risk measures. In Appendix 2 we derive the granularity approximation for the VaR of an heterogeneous portfolio and relate it to the risk measures of the different homogeneous subpopulations.

4 Examples

This section provides the closed form expressions of the GA for the examples introduced in Section 2. We first consider single factor models, in which the GA formula is greatly simplified, then models with multiple factor.

4.1 Single Risk Factor Model

In a single factor model, the factor can generally be identified with the expected individual risk:

$$m(F) = F. \tag{4.1}$$

Then, $VaR_\infty(\alpha)$ is the α-quantile of factor F and g_∞ is its density function. The granularity adjustment of the VaR becomes:

$$GA(\alpha) = -\frac{1}{2}\left\{\frac{d\log g_\infty}{dF}[VaR_\infty(\alpha)]\sigma^2[VaR_\infty(\alpha)] + \frac{d\sigma^2}{dF}[VaR_\infty(\alpha)]\right\}. \tag{4.2}$$

This formula has been initially derived by Wilde (2001) [see also Martin, Wilde (2002), Gordy (2003, 2004)], based on the local analysis of the VaR [Gouriéroux, Laurent, Scaillet (2000)] and Expected Shortfall [Tasche (2000)]. [6]

Example 4.1: Linear Single Risk Factor Model [Gordy (2004)]

In the linear model $y_i = F + u_i$, with $F \sim N(\mu, \eta^2)$ and $u_i \sim N(0, \sigma^2)$, we have $m(F) = F$, $\sigma^2(F) = \sigma^2$, $g_\infty(F) = \frac{1}{\eta}\varphi\left(\frac{F-\mu}{\eta}\right)$, and $VaR_\infty(\alpha) = \mu + \eta\Phi^{-1}(\alpha)$, where φ (resp. Φ) is the density function (resp. cumulative distribution function) of the standard normal distribution. We deduce:

$$GA(\alpha) = -\frac{\sigma^2}{2}\frac{d\log g_\infty}{dF}[VaR_\infty(\alpha)] = \frac{\sigma^2}{2\eta}\Phi^{-1}(\alpha).$$

In this simple Gaussian framework, the quantile $VaR_n(\alpha)$ is known in closed form, and the above GA corresponds to the first-order term in a Taylor expansion of $VaR_n(\alpha)$ w.r.t. $1/n$. Indeed, we have $W_n/n \sim N\left(\mu, \eta^2 + \sigma^2/n\right)$ and:

$$VaR_n(\alpha) = \mu + \sqrt{\eta^2 + \frac{\sigma^2}{n}}\Phi^{-1}(\alpha) = \mu + \eta\Phi^{-1}(\alpha) + \frac{1}{n}\frac{\sigma^2}{2\eta}\Phi^{-1}(\alpha) + o(1/n).$$

As expected the GA is positive for large α, more precisely for $\alpha > 0.5$. The GA increases when the idiosyncratic risk increases, that is, when σ^2 increases. Moreover, the GA is a decreasing function of η, which means that the adjustment is smaller, when systematic risk increases.

Example 4.2: Static RFM for Default

Let us assume that the individual risks follow independent Bernoulli distributions conditional on factor F:

$$y_i \sim \mathcal{B}(1, F),$$

where $\mathcal{B}(1, p)$ denotes a Bernoulli distribution with probability p. This is the well-known model with stochastic probability of default, often called reduced form model or stochastic intensity model in the Credit Risk literature. In this case W_n/n is the default frequency in the portfolio.

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[6] When the factor F is not identified with the conditional mean $m(F)$, but the function $m(.)$ is increasing, formula (4.2) becomes:

$$GA(\alpha) = -\frac{1}{2}\left\{\frac{1}{\tilde{g}_\infty(F)}\frac{d}{dF}\left[\frac{\sigma^2(F)\tilde{g}_\infty(F)}{\frac{dm}{dF}(F)}\right]\right\}_{F=m^{-1}(VaR_\infty(\alpha))}, \quad (4.3)$$

where $\tilde{g}_\infty$ is the density of F (see Appendix 3 for the derivation).

It also corresponds to the portfolio loss given default, if the loans have a unitary nominal and a zero recovery rate. In this model we have $m(F) = F$, $\sigma^2(F) = F(1-F)$, and we deduce:

$$GA(\alpha) = -\frac{1}{2}\left\{\frac{d\log g_\infty}{dF}[VaR_\infty(\alpha)]VaR_\infty(\alpha)[1-VaR_\infty(\alpha)] + 1 - 2VaR_\infty(\alpha)\right\}. \quad (4.4)$$

This formula appears for instance in Rau-Bredow (2005).

Different specifications have been considered in the literature for stochastic intensity F. [7] Let us assume that there exists an increasing transformation A, say, from $(-\infty, +\infty)$ to (0,1) such that:

$$A^{-1}(F) \sim N(\mu, \eta^2). \quad (4.5)$$

We get a logit (resp. probit) normal specification, when A is the cumulative distribution function of the the logistic distribution (resp. the standard normal distribution). A logit specification is used in CreditPortfolioView by Mc Kinsey, a probit specification is proposed in KMV/Moody's and CreditMetrics. Let us denote $a(y) = \dfrac{dA(y)}{dy}$ the associated derivative. We have:

$$\begin{aligned} VaR_\infty(\alpha) &= A[\mu + \eta\Phi^{-1}(\alpha)], \\ \frac{d\log g_\infty}{dF}[VaR_\infty(\alpha)] &= -\frac{1}{a[\mu+\eta\Phi^{-1}(\alpha)]}\left(\frac{\Phi^{-1}(\alpha)}{\eta} + \frac{d\log a}{dy}[\mu+\eta\Phi^{-1}(\alpha)]\right). \end{aligned} \quad (4.6)$$

i) In the logit normal reduced form, the transformation $A^{-1}(F) = \log[F/(1-F)]$ corresponds to the log of an odd ratio, and the formula for the GA simplifies considerably:

$$VaR_\infty(\alpha) = \frac{1}{1+\exp[-\mu-\eta\Phi^{-1}(\alpha)]}, \quad GA(\alpha) = \frac{1}{2\eta}\Phi^{-1}(\alpha).$$

In particular, the GA doesn't depend on parameter μ.

ii) Let us now consider the structural Merton (1974) - Vasicek (1991) model [see Example 2.2]. This model can be written in terms of two structural parameters, that are the unconditional

[7] Gordy (2004) and Gordy, Lutkebohmert (2007) derive the GA in the CreditRisk+ model [Credit Suisse Financial Products (1997)], which has been the basis for the granularity adjustment proposed in the New Basel Capital Accord [see BCBS (2001, Chapter 8) and Wilde (2001)]. The CreditRisk+ model has some limitations. First, it assumes that the stochastic probability of default F follows a gamma distribution, that admits values of default probability larger than 1. Second, it assumes a constant expected loss given default. We present a multi-factor model with stochastic default probability and expected loss given default in Section 6.

probability of default PD and the asset correlation ρ, such as:

$$y_i = \mathbb{1}_{-\Phi^{-1}(PD) + \sqrt{\rho}F^* + \sqrt{1-\rho}u_i^* < 0},$$

where F^* and u_i^*, for $i = 1, \cdots, n$, are independent standard Gaussian variables. The structural factor F^* is distinguished from the reduced form factor F which is the stochastic probability of default. They are related by:

$$\Phi^{-1}(F) = \frac{\Phi^{-1}(PD)}{\sqrt{1-\rho}} - \frac{\sqrt{\rho}}{\sqrt{1-\rho}}F^*.$$

From (4.5) we deduce that:

$$\mu = \frac{\Phi^{-1}(PD)}{\sqrt{1-\rho}}, \quad \eta = \sqrt{\frac{\rho}{1-\rho}}. \tag{4.7}$$

Thus, from (4.4)-(4.7) we deduce the CSA VaR [Vasicek (1991)]:

$$VaR_\infty(\alpha) = \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right), \tag{4.8}$$

and the granularity adjustment (see Appendix 4):

$$GA(\alpha) = \frac{1}{2}\left\{\frac{\sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(\alpha) - \Phi^{-1}\left[VaR_\infty(\alpha)\right]}{\phi\left(\Phi^{-1}\left[VaR_\infty(\alpha)\right]\right)}VaR_\infty(\alpha)[1 - VaR_\infty(\alpha)] + 2VaR_\infty(\alpha) - 1\right\}. \tag{4.9}$$

Equation (4.9) is similar to formula (2.17) in Emmer, Tasche (2005) [see also Gordy, Marrone (2010), equation (5)], but is written in a way that shows how the GA depends on the unconditional probability of default PD and asset correlation ρ. This dependence occurs through the term $\sqrt{(1-\rho)/\rho}$ and the CSA quantile $VaR_\infty(\alpha)$.

In Figure 1 we display the CSA quantile $VaR_\infty(\alpha)$ and the granularity adjustment per contract $\frac{1}{n}GA(\alpha)$ as functions of asset correlation ρ and for different values of the unconditional probability of default, that are $PD = 0.5\%$, 1%, 5%, and 20%, respectively. These values of PD are representative for the default probabilities of a firm with rating BBB, BB, B and C, respectively, in the rating system by S&P. The confidence level is $\alpha = 0.99$ and the portfolio size is $n = 1000$.

[Insert Figure 1: CSA CreditVaR and granularity adjustment as functions of the asset correlation in the Merton-Vasicek model.]

The CSA VaR is monotone increasing w.r.t. asset correlation ρ when the probability of default is such that $PD \geq 1-\alpha$; for $PD < 1-\alpha$, the CSA VaR is increasing w.r.t. ρ up to a maximum and then converges to zero as ρ approaches 1. In the interval of asset correlation values $\rho \in [0.12, 0.24]$ considered for obligors in the Basel 2 regulation [see BCBS(2001)], the CSA VaR is about 0.05 for obligors with $PD = 1\%$, while the GA per contract is about 0.005 for $n = 1000$ (and 0.05 for $n = 100$). Thus, the magnitude of the GA can be significant w.r.t. the CSA VaR. The granularity adjustment is decreasing w.r.t. asset correlation ρ, when ρ is not close to 1.

In Figure 2 we display the CSA quantile $VaR_\infty(\alpha)$ and the granularity adjustment per contract $\frac{1}{n}GA(\alpha)$ as functions of probability of default PD and for different values of the asset correlation, that are $\rho = 0.05$, 0.12, 0.24, and 0.50, respectively.

[Insert Figure 2: CSA CreditVaR and granularity adjustment as functions of the unconditional probability of default in the Merton-Vasicek model.]

The CSA VaR is monotone increasing w.r.t. the probability of default PD. The granularity adjustment features an inverse-U shape. The maximum GA occurs for values of PD corresponding to speculative grade ratings, when ρ is between 0.12 and 0.24.

Example 4.3: Linear Static RFM with Beta Heterogeneity

Let us consider the model of Example 2.4. We have $y_i = \beta_i F + v_i$, where $F \sim N(\mu, \eta^2)$, $v_i \sim N(0, \sigma^2)$, and $\beta_i \sim N(1, \gamma^2)$, with all these variables independent. Due to a problem of factor identification, the mean of β_i can always be fixed to 1. This also facilitates the comparison with the model with constant beta of Example 4.1. We get $m(F) = F$, $\sigma^2(F) = \sigma^2 + \gamma^2 F^2$, and:

$$GA(\alpha) = \frac{\sigma^2}{2\eta}\Phi^{-1}(\alpha) + \gamma^2 \left[VaR_\infty(\alpha)^2 \frac{\Phi^{-1}(\alpha)}{2\eta} - VaR_\infty(\alpha) \right], \tag{4.10}$$

where $VaR_\infty(\alpha) = \mu + \eta\Phi^{-1}(\alpha)$. Thus, the CSA risk measure $VaR_\infty(\alpha)$ is computed in the homogenous model with factor sensitivity $\beta = 1$. The granularity adjustment accounts for beta heterogeneity in the portfolio through the variance γ^2 of the heterogeneity distribution. More precisely, the first term in the RHS of (4.10) is the GA already derived in Example 4.1, whereas the second term is specific of the beta heterogeneity.

4.2 Multiple Risk Factor Model

Example 4.4: Stochastic Drift and Volatility Model

Let us assume that $y_i \sim N(F_1, \exp F_2)$, conditional on the bivariate factor $F = (F_1, F_2)'$, and that

$$F \sim N \left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \eta_1^2 & \rho\eta_1\eta_2 \\ \rho\eta_1\eta_2 & \eta_2^2 \end{pmatrix} \right].$$

This type of stochastic volatility model is standard for modelling the dynamic of asset returns, or equivalently of opposite asset returns.

We can introduce the regression equation:

$$F_2 = \mu_2 + \frac{\rho\eta_2}{\eta_1}(F_1 - \mu_1) + v,$$

where v is independent of F_1, with Gaussian distribution $N[0, \eta_2^2(1-\rho^2)]$. We have $m(F) = F_1$, $\sigma^2(F) = \exp(F_2)$, and

$$\begin{aligned} E[\sigma^2(F)|m(F)] &= E[\exp F_2|F_1] = E[\exp\{\mu_2 + \frac{\rho\eta_2}{\eta_1}(F_1 - \mu_1) + v\}|F_1] \\ &= \exp\left[\mu_2 + \frac{\rho\eta_2}{\eta_1}(F_1 - \mu_1)\right] E[\exp(v)] \\ &= \exp\left[\mu_2 + \frac{\rho\eta_2}{\eta_1}(F_1 - \mu_1) + \frac{\eta_2^2(1-\rho^2)}{2}\right]. \end{aligned}$$

In particular:

$$\frac{d}{dw} \log E\left[\sigma^2(F)|m(F) = w\right] = \frac{d}{dF_1} \log E\left[F_2|F_1\right] = \frac{\rho\eta_2}{\eta_1}.$$

When y_i is the opposite of an asset return, a positive value $\rho > 0$, i.e. a negative correlation between return and volatility, can represent a leverage effect. From Proposition 1, we deduce that:

$$GA(\alpha) = \frac{1}{2\eta_1}[\Phi^{-1}(\alpha) - \rho\eta_2] \exp[\mu_2 + \eta_2^2/2] \exp\left[\rho\eta_2\Phi^{-1}(\alpha) + \frac{\rho^2\eta_2^2}{2}\right].$$

The GA of the linear single-factor RFM (see Example 4.1) is obtained when either factor F_2 is constant ($\eta_2 = 0$), or factors F_1 and F_2 are independent ($\rho = 0$), by noting that $E[\exp F_2] = \exp[\mu_2 + \eta_2^2/2]$.

5 Dynamic Risk Factor Model (DRFM)

The static risk factor model implicitly assumes that the past observations are not informative to predict the future risk. In dynamic factor models, the VaR becomes a function of the available information. This conditional VaR has to account for the unobservability of the current and lagged factor values. We show in this section that factor unobservability implies an additional GA for the VaR. Despite this further layer of complexity in dynamic models, the granularity principle becomes even more useful compared to the static framework. Indeed, the conditional cdf of the portfolio value at date t involves an integral that can reach dimension $(t+1)dim(F) + n\, dim(u) - 1$, which now depends on t, due to the integration w.r.t. the factor path.

5.1 The Model

Dynamic features can easily be introduced in the following way:

i) We still assume a static relationship between the individual risks and the systematic factors. This relationship is given by the static measurement equations:

$$y_{it} = c(F_t, u_{it}), \tag{5.1}$$

where the idiosyncratic risks $(u_{i,t})$ are independent, identically distributed across individuals and dates, and independent of the factor process (F_t).

ii) Then, we allow for factor dynamic. The factor process (F_t) is Markov with transition pdf $g(f_t|f_{t-1})$, say. Thus, all the dynamics of individual risks pass by means of the factor dynamic.

Let us now consider the future portfolio risk per individual asset defined by $W_{n,t+1}/n = \frac{1}{n}\sum_{i=1}^{n} y_{i,t+1}$. The (conditional) VaR at horizon 1 is defined by the equation:

$$P[W_{n,t+1}/n < VaR_{n,t}(\alpha)|I_{n,t}] = \alpha, \tag{5.2}$$

where the available information $I_{n,t}$ includes all current and past individual risks $y_{i,t}, y_{i,t-1}, \ldots$, for $i = 1, \ldots, n$, but not the current and past factor values. The conditional quantile $VaR_{n,t}(\alpha)$ depends on date t through the information $I_{n,t}$.

5.2 Granularity Adjustment

i) Asymptotic expansion of the portfolio risk

Let us first perform an asymptotic expansion of the portfolio risk. By the cross-sectional CLT applied conditional on the factor path, we have:

$$W_{n,t+1}/n = m(F_{t+1}) + \sigma(F_{t+1})\frac{X_{t+1}}{\sqrt{n}} + O(1/n),$$

where the variable X_{t+1} is standard normal, independent of the factor process, and $O(1/n)$ denotes a term of order $1/n$, which is zero-mean conditional on $F_{t+1}, F_t, \cdots$. The functions $m(.)$ and $\sigma^2(.)$ are defined analogously as in (3.2)-(3.3), and depend on F_{t+1} only by the static measurement equations (5.1).

In order to compute the conditional cdf of $W_{n,t+1}/n$, it is useful to reintroduce the current factor value in the conditioning set through the law of iterated expectation. We have:

$$\begin{aligned} P[W_{n,t+1}/n < y|I_{n,t}] &= E[P(W_{n,t+1}/n < y|F_t, I_{n,t})|I_{n,t}] \\ &= E[P(W_{n,t+1}/n < y|F_t)|I_{n,t}] \\ &= E[P(m(F_{t+1}) + \sigma(F_{t+1})\frac{X_{t+1}}{\sqrt{n}} + O(1/n) < y|F_t)|I_{n,t}] \\ &= E[a(y, \frac{X_{t+1}}{\sqrt{n}} + O(1/n); F_t)|I_{n,t}], \end{aligned} \tag{5.3}$$

where function a is defined by:

$$a(y, \varepsilon; f_t) = P[m(F_{t+1}) + \sigma(F_{t+1})\varepsilon < y|F_t = f_t]. \tag{5.4}$$

ii) Cross-sectional approximation of the factor

Function a depends on the unobserved factor value $F_t = f_t$, and we have first to explain how this value can be approximated from observed individual variables. For this purpose, let us denote by $h(y_{i,t}|f_t)$ the conditional density of $y_{i,t}$ given $F_t = f_t$, deduced from model (5.1), and define the **cross-sectional maximum likelihood** approximation of f_t given by:

$$\hat{f}_{n,t} = \arg\max_{f_t} \sum_{i=1}^{n} \log h(y_{i,t}|f_t). \tag{5.5}$$

The factor value f_t is treated as an unknown parameter in the cross-sectional conditional model at date t and is approximated by the maximum likelihood principle. Approximation $\hat{f}_{nt}$ is a function of the current individual observations and hence of the available information $I_{n,t}$.

iii) Granularity Adjustment for factor prediction

It might seem natural to replace the unobserved factor value f_t by its cross-sectional approximation $\hat{f}_{n,t}$ in the expression of function a, and then to use the GA of the static model in Proposition 1, for the distribution of F_{t+1} given $F_t = \hat{f}_{n,t}$. However, replacing f_t by $\hat{f}_{n,t}$ implies an approximation error. It has been proved that this error is of order $1/n$, that is, the same order expected for the GA. More precisely, we have the following result which is given in the single factor framework for expository purpose [Gagliardini, Gouriéroux (2010a), Corollary 5.3]:

Proposition 2: *Let us consider a dynamic single factor model. For a large homogenous portfolio, the conditional distribution of F_t given $I_{n,t}$ is approximately normal at order $1/n$:*

$$N\left(\hat{f}_{n,t} + \frac{1}{n}\mu_{n,t}, \frac{1}{n}J_{n,t}^{-1}\right),$$

where:

$$\begin{aligned}
\mu_{n,t} &= J_{n,t}^{-1}\frac{\partial \log g}{\partial f_t}(\hat{f}_{n,t}|\hat{f}_{n,t-1}) + \frac{1}{2}J_{n,t}^{-2}K_{n,t}, \\
J_{n,t} &= -\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2 \log h}{\partial f_t^2}(y_{i,t}|\hat{f}_{n,t}), \\
K_{n,t} &= \frac{1}{n}\sum_{i=1}^{n}\frac{\partial^3 \log h}{\partial f_t^3}(y_{i,t}|\hat{f}_{n,t}).
\end{aligned}$$

Proposition 2 gives an approximation of the filtering distribution of factor F_t given the information $I_{n,t}$. Both mean and variance are approximated at order $1/n$ to apply an Ito's type correction. The approximation involves four summary statistics, which are the cross-sectional maximum likelihood approximations $\hat{f}_{n,t}$ and $\hat{f}_{n,t-1}$, the Fisher information $J_{n,t}$ for approximating the factor in the cross-section at date t, and the statistic $K_{n,t}$ involved in the bias adjustment.

iii) Expansion of the cdf of portfolio risk

From equation (5.3) and Proposition 2, the conditional cdf of the portfolio risk can be written as:

$$P[W_{n,t+1}/n < y|I_{n,t}] = E\left[a\left(y, \frac{X_{t+1}}{\sqrt{n}} + O(1/n); \hat{f}_{n,t} + \frac{1}{n}\mu_{n,t} + \frac{1}{\sqrt{n}}J_{n,t}^{-1/2}X_t^*\right)|I_{n,t}\right] + o(1/n),$$

where X_t^* is a standard Gaussian variable independent of X_{t+1} and $O(1/n)$, of the factor path and of the available information [8].

Then, we can expand the expression above with respect to n, up to order $1/n$. By noting that $\hat{f}_{n,t}$, $\mu_{n,t}$, $J_{n,t}$ are functions of the available information and that $E[X_{t+1}] = E[X_t^*] = E[O(1/n)] = 0$, $E[X_{t+1}X_t^*] = 0$, $E[X_{t+1}^2] = E[(X_t^*)^2] = 1$, we get:

$$\begin{aligned} P[W_{n,t+1}/n < y | I_{n,t}] &= a(y,0;\hat{f}_{n,t}) + \frac{1}{n}\frac{\partial a(y,0;\hat{f}_{nt})}{\partial f_t}\mu_{nt} \\ &\quad + \frac{1}{2n}\left[J_{n,t}^{-1}\frac{\partial^2 a(y,0;\hat{f}_{n,t})}{\partial f_t^2} + \frac{\partial^2 a(y,0,\hat{f}_{nt})}{\partial \varepsilon^2}\right] + o(1/n). \end{aligned}$$

The CSA conditional cdf of the portfolio risk is $a(y,0;\hat{f}_{n,t})$, where:

$$a(y,0;f_t) = P[m(F_{t+1}) < y | F_t = f_t]. \tag{5.6}$$

It corresponds to the conditional cdf of $m(F_{t+1})$ given $F_t = f_t$, where the unobservable factor value f_t is replaced by its cross-sectional approximation $\hat{f}_{n,t}$. The GA for the cdf is the sum of the following components:

i) The granularity adjustment for the conditional cdf with known F_t equal to $\hat{f}_{n,t}$ is

$$\frac{1}{2n}\frac{\partial^2 a(y,0;\hat{f}_{n,t})}{\partial \varepsilon^2}. \tag{5.7}$$

The second-order derivative of function $a(y,\varepsilon;f_t)$ w.r.t. ε at $\varepsilon = 0$ can be computed by using Lemma a.1 in Appendix 1, which yields:

$$\frac{\partial^2 a(y,0;f_t)}{\partial \varepsilon^2} = \frac{d}{dy}\left\{g_\infty(y;f_t)E[\sigma^2(F_{t+1})|m(F_{t+1}) = y, F_t = f_t]\right\},$$

where $g_\infty(y;f_t)$ denotes the pdf of $m(F_{t+1})$ conditional on $F_t = f_t$.

ii) The granularity adjustment for filtering is

$$\frac{\partial a(y,0;\hat{f}_{n,t})}{\partial f_t}\mu_{nt} + \frac{1}{2}J_{n,t}^{-1}\frac{\partial^2 a(y,0;\hat{f}_{nt})}{\partial f_t^2}. \tag{5.8}$$

It involves the first- and second-order derivatives of the CSA cdf w.r.t. the conditioning factor value.

[8] The independence between X_t^* and X_{t+1} is due to the fact that X_t^* simply represents the numerical approximation of the filtering distribution of F_t given $I_{n,t}$ and is not related to the stochastic features of the observations at $t+1$ [see Gagliardini, Gouriéroux (2010a)].

Due to the independence between variables X_{t+1} and X_t^*, there is no cross GA.

iv) Granularity Adjustment for the VaR

The CSA cdf is used to define the CSA risk measure $VaR_\infty(\alpha; \hat{f}_{n,t})$ through the condition:

$$P\left[m(F_{t+1}) < VaR_\infty(\alpha; \hat{f}_{n,t}) | F_t = \hat{f}_{n,t}\right] = \alpha.$$

The CSA VaR depends on the current information through the cross-sectional approximation of the factor value $\hat{f}_{n,t}$ only. The GA for the (conditional) VaR is directly deduced from the GA of the (conditional) cdf by applying the Bahadur's expansion [Bahadur (1966); see Lemma a.3 in Appendix 1]. We get the next Proposition:

Proposition 3: *In a dynamic RFM the (conditional) VaR is such that:*

$$VaR_{n,t}(\alpha) = VaR_\infty(\alpha; \hat{f}_{n,t}) + \frac{1}{n}\left[GA_{risk,t}(\alpha) + GA_{filt,t}(\alpha)\right] + o(1/n),$$

where:

$$\begin{aligned} GA_{risk,t}(\alpha) \quad = \quad & -\frac{1}{2}\left\{\frac{\partial \log g_\infty(w; \hat{f}_{nt})}{\partial w} E[\sigma^2(F_{t+1}) | m(F_{t+1}) = w, F_t = \hat{f}_{n,t}]\right. \\ & \left. + \frac{\partial}{\partial w} E[\sigma^2(F_{t+1}) | m(F_{t+1}) = w, F_t = \hat{f}_{n,t}]\right\}_{w = VaR_\infty(\alpha; \hat{f}_{n,t})}, \end{aligned}$$

and:

$$\begin{aligned} GA_{filt,t}(\alpha) \quad = \quad & -\frac{1}{g_\infty[VaR_\infty(\alpha; \hat{f}_{n,t}); \hat{f}_{n,t}]}\left\{\frac{\partial a[VaR_\infty(\alpha, \hat{f}_{nt}), 0; \hat{f}_{nt}]}{\partial f_t}\mu_{nt}\right. \\ & \left. + \frac{1}{2} J_{n,t}^{-1} \frac{\partial^2 a[VaR_\infty(\alpha; \hat{f}_{nt}), 0; \hat{f}_{nt}]}{\partial f_t^2}\right\}, \end{aligned}$$

and where $g_\infty(.; f_t)$ [resp. $a(., 0; f_t)$ and $VaR_\infty(.; f_t)$] denotes the pdf (resp. the cdf and quantile) of $m(F_{t+1})$ conditional on $F_t = f_t$.

Thus, the GA for the conditional VaR is the sum of two components. The first one $GA_{risk,t}(\alpha)$ is the analogue of the GA in the static factor model (see Proposition 1). However, the distribution of $m(F_{t+1})$ and $\sigma^2(F_{t+1})$ is now conditional on $F_t = f_t$, and the unobservable factor value f_t is replaced by its cross-sectional approximation $\hat{f}_{n,t}$. The second component $GA_{filt,t}(\alpha)$ is due to the filtering of the unobservable factor value, and involves first- and second-order derivatives of the CSA cdf w.r.t. the conditioning factor value.

5.3 Linear RFM with AR(1) factor

As an illustration, let us consider the model given by:

$$y_{i,t} = F_t + u_{i,t}, \quad i = 1, \dots, n,$$

where:

$$F_t = \mu + \rho(F_{t-1} - \mu) + v_t,$$

and $u_{i,t}$, v_t are independent, with $u_{it} \sim IIN(0, \sigma^2)$, $v_t \sim IIN(0, \eta^2)$. In this Gaussian framework the (conditional) VaR can be computed explicitly, which allows for a comparison with the granularity approximation in order to assess the accuracy of the two GA components and their (relative) magnitude.

The individual observations can be summarized by their cross-sectional averages [9] and we have:

$$\begin{cases} \bar{y}_{n,t+1} &= F_{t+1} + \bar{u}_{n,t+1}, \\ F_{t+1} &= \mu + \rho(F_t - \mu) + v_{t+1}. \end{cases} \tag{5.9}$$

This implies:

$$\bar{y}_{n,t+1} = \mu + \frac{1}{1-\rho L} v_{t+1} + \bar{u}_{n,t+1} = \mu + \frac{1}{1-\rho L}(v_{t+1} + \bar{u}_{n,t+1} - \rho \bar{u}_{n,t}),$$

where L denotes the lag operator. The process $Z_{t+1} = v_{t+1} + \bar{u}_{n,t+1} - \rho \bar{u}_{n,t}$ is a Gaussian MA(1) process that can be written as $Z_{t+1} = \varepsilon_{t+1} - \theta_n \varepsilon_t$, where the variables ε_t are $IIN(0, \gamma_n^2)$, say, and $|\theta_n| < 1$. The new parameters θ_n and γ_n are deduced from the expressions of the variance and autocovariance at lag 1 of process (Z_t). They satisfy:

$$\begin{cases} \eta^2 + \dfrac{\sigma^2}{n}(1 + \rho^2) = \gamma_n^2(1 + \theta_n^2), \\ \rho \dfrac{\sigma^2}{n} = \theta_n \gamma_n^2. \end{cases} \tag{5.10}$$

Hence:

$$\theta_n = \frac{b_n - \sqrt{b_n^2 - 4\rho^2}}{2\rho}, \quad \gamma_n^2 = \frac{\rho\sigma^2}{n\theta_n}, \tag{5.11}$$

[9] By writing the likelihood of the model, it is seen that the cross-sectional averages are sufficient statistics.

where $b_n = 1 + \rho^2 + n\frac{\eta^2}{\sigma^2}$ and we have selected the root θ_n such that $|\theta_n| < 1$. Therefore, variable $\bar{y}_{n,t+1}$ follows a Gaussian ARMA(1,1) process, and we can write:

$$\begin{aligned}
& \bar{y}_{n,t+1} &=& \mu + \frac{1-\theta_n L}{1-\rho L}\varepsilon_{t+1} \\
\iff & \bar{y}_{n,t+1} &=& \mu + \left(1 - \frac{1-\rho L}{1-\theta_n L}\right)(\bar{y}_{n,t+1} - \mu) + \varepsilon_{t+1} \\
\iff & \bar{y}_{n,t+1} &=& \mu + \frac{\rho - \theta_n}{1-\theta_n L}(\bar{y}_{n,t} - \mu) + \varepsilon_{t+1} \\
\iff & \bar{y}_{n,t+1} &=& \mu + (\rho - \theta_n)\sum_{j=0}^{\infty}\theta_n^j(\bar{y}_{n,t-j} - \mu) + \varepsilon_{t+1}.
\end{aligned}$$

We deduce that the conditional distribution of $W_{n,t+1}/n = \bar{y}_{n,t+1}$ given $I_{n,t}$ is Gaussian with mean $\mu + (\rho - \theta_n)\sum_{j=0}^{\infty}\theta_n^j(\bar{y}_{n,t-j} - \mu)$ and variance γ_n^2.

Proposition 4: *In the linear RFM with AR(1) Gaussian factor, the conditional VaR is given by:*

$$VaR_{n,t}(\alpha) = \mu + (\rho - \theta_n)\sum_{j=0}^{\infty}\theta_n^j(\bar{y}_{n,t-j} - \mu) + \gamma_n\Phi^{-1}(\alpha),$$

where θ_n and γ_n are given in (5.11).

Thus, the conditional VaR depends on the information $I_{n,t}$ through a weighted sum of current and lagged cross-sectional individual risks averages. The weights decay geometrically with the lag as powers of parameter θ_n. Moreover, the information $I_{n,t}$ impacts the VaR uniformly in the risk level α.

Let us now derive the expansion of $VaR_{n,t}(\alpha)$ at order $1/n$ for large n. From (5.11), the expansions of parameters θ_n and γ_n are:

$$\theta_n = \frac{\rho\sigma^2}{\eta^2 n} + o(1/n), \quad \gamma_n = \eta + \frac{\sigma^2}{2n\eta}(1+\rho^2) + o(1/n).$$

As $n \to \infty$, the MA parameter θ_n converges to zero and the variance of the shocks γ_n^2 converges to η^2. Hence, the ARMA(1,1) process of the cross-sectional averages $\bar{y}_{n,t}$ approaches the AR(1) factor process (F_t) as expected. By plugging the expansions for θ_n and γ_n into the expression of $VaR_{n,t}(\alpha)$ in Proposition 4, we get:

$$\begin{aligned}
VaR_{n,t}(\alpha) &=& \mu + \rho(\bar{y}_{n,t} - \mu) + \eta\Phi^{-1}(\alpha) \\
&& + \frac{1}{n}\left\{\frac{\sigma^2}{2\eta}(1+\rho^2)\Phi^{-1}(\alpha) - \frac{\rho\sigma^2}{\eta^2}\left[(\bar{y}_{n,t} - \mu) - \rho(\bar{y}_{n,t-1} - \mu)\right]\right\} + o(1/n).
\end{aligned}$$

The first row on the RHS provides the CSA VaR:

$$VaR_{\infty}(\alpha; \hat{f}_{n,t}) = \mu + \rho(\bar{y}_{n,t} - \mu) + \eta\Phi^{-1}(\alpha). \tag{5.12}$$

which depends on the information through the cross-sectional maximum likelihood approximation of the factor $\hat{f}_{n,t} = \bar{y}_{n,t}$. The CSA VaR is the quantile of the normal distribution with mean $\mu + \rho(\bar{y}_{n,t} - \mu)$ and variance η^2, that is the conditional distribution of F_{t+1} given $F_t = \bar{y}_{n,t}$. The GA involves the information $I_{n,t}$ through the current and lagged cross-sectional averages $\bar{y}_{n,t}$ and $\bar{y}_{n,t-1}$. The other lagged values $\bar{y}_{n,t-j}$ for $j \geq 2$ are irrelevant at order $o(1/n)$.

Let us now identify the risk and filtering GA components. We have:

$$a(y, 0; f_t) = P(F_{t+1} < y | F_t = f_t) = \Phi\left(\frac{y - \mu - \rho(f_t - \mu)}{\eta}\right).$$

We deduce:

$$g_{\infty}(y; f_t) = \frac{\partial a(y, 0; f_t)}{\partial y} = \frac{1}{\eta}\varphi\left[\frac{y - \mu - \rho(f_t - \mu)}{\eta}\right],$$

and:

$$\begin{aligned}
\frac{\partial a(y, 0; f_t)}{\partial f_t} &= -\frac{\rho}{\eta}\varphi\left[\frac{y - \mu - \rho(f_t - \mu)}{\eta}\right], \\
\frac{\partial^2 a(y, 0; f_t)}{\partial f_t^2} &= \frac{\rho^2}{\eta^2}\left[\frac{y - \mu - \rho(f_t - \mu)}{\eta}\right]\varphi\left(\frac{y - \mu - \rho(f_t - \mu)}{\eta}\right).
\end{aligned}$$

Moreover, the statistics involved in the approximate filtering distribution of F_t given $I_{n,t}$ (see Proposition 2) are $\mu_{n,t} = -\frac{\sigma^2}{\eta^2}\left[(\bar{y}_{n,t} - \mu) - \rho(\bar{y}_{n,t-1} - \mu)\right]$, $J_{n,t} = 1/\sigma^2$ and $K_{n,t} = 0$. From Proposition 2 and equation (5.12), we get:

$$\begin{aligned}
GA_{risk}(\alpha) &= \frac{\sigma^2}{2\eta}\Phi^{-1}(\alpha), \\
GA_{filt,t}(\alpha) &= \frac{\sigma^2\rho^2}{2\eta}\Phi^{-1}(\alpha) - \frac{\rho\sigma^2}{\eta^2}\left[(\bar{y}_{n,t} - \mu) - \rho(\bar{y}_{n,t-1} - \mu)\right].
\end{aligned}$$

The GA for risk is the same as in the static model for $\rho = 0$ (see Example 4.1), since in this Gaussian framework the current factor f_t impacts the conditional distribution of $m(F_{t+1}) = F_{t+1}$ given $F_t = f_t$ through the mean only, and $\sigma^2(F_{t+1}) = \sigma^2$ is constant. The GA for filtering depends on both the risk level α and the information through $(\bar{y}_{n,t} - \mu) - \rho(\bar{y}_{n,t-1} - \mu)$. By the latter effect, $GA_{filt,t}(\alpha)$ can take any sign. Moreover, this term induces a stabilization effect on the dynamics of the GA VaR compared to the CSA VaR. To see this, let us assume $\rho > 0$ and suppose there is a large upward aggregate shock on the individual risks at date t, such that $\bar{y}_{n,t} - \mu$ is positive and (much) larger than $\rho(\bar{y}_{n,t-1} - \mu)$. The CSA VaR in (5.12) reacts linearly to the shock and features a sharp increase. Since $(\bar{y}_{n,t} - \mu) - \rho(\bar{y}_{n,t-1} - \mu) > 0$, the GA term for filtering is negative and reduces the reaction of the VaR.

In Figure 3 we display the patterns of the true, CSA and GA VaR curves as a function of the risk level for a specific choice of parameters.

[Insert Figure 3: The VaR as a function of the risk level in the linear RFM with AR(1) factor.]

The mean and the autoregressive coefficient of the factor are $\mu = 0$ and $\rho = 0.5$, respectively. The idiosyncratic and systematic variance parameters σ^2 and η^2 are selected in order to imply an unconditional standard deviation of the individual risks $\sqrt{\eta^2/(1-\rho^2) + \sigma^2} = 0.15$, and an unconditional correlation between individual risks $\dfrac{\eta^2/(1-\rho^2)}{\sigma^2 + \eta^2/(1-\rho^2)} = 0.10$. The portfolio size is $n = 100$. The available information $I_{n,t}$ is such that $\bar{y}_{n,t-j} = \mu = 0$ for all lags $j \geq 2$, and we consider four different cases concerning the current and the most recent lagged cross-sectional averages, $\bar{y}_{n,t}$ and $\bar{y}_{n,t-1}$, respectively. Let us first assume $\bar{y}_{n,t} = \bar{y}_{n,t-1} = 0$ (upper-left Panel), that is, both cross-sectional averages are equal to the unconditional mean. As expected, all VaR curves are increasing w.r.t. the confidence level. The true VaR is about 0.10 at confidence level 99%. The CSA VaR underestimates the true VaR (that is, underestimates the risk) by about 0.01 uniformly in the risk level. The GA for risk corrects most of this bias and dominates the GA for filtering. The situation is different when $\bar{y}_{n,t} = -0.30$ and $\bar{y}_{n,t-1} = 0$ (upper-right Panel), that is, when we have a downward aggregate shock in risk of two standard deviations at date t. The CSA VaR underestimates the true VaR by about 0.02. The GA for risk corrects only a rather small part of this bias, while including the GA for filtering allows for a quite accurate approximation. The GA for filtering is about five times larger than the GA for risk. When $\bar{y}_{n,t} = 0.30$ and $\bar{y}_{n,t-1} = 0$ (lower-left Panel), there is a large upward aggregate shock in risk at date t, and the CSA VaR overestimates the true VaR (that is, overestimates the risk). The GA correction for risk further increases the VaR and the bias, while including the GA correction yields a good approximation of the true VaR. The results are similar in the case $\bar{y}_{n,t} = 0.30$ and $\bar{y}_{n,t-1} = 0.30$ (lower-right Panel), that is, in case of a persistent downward aggregate shock in risk. Finally, by comparing the four panels in Figure 3, it is seen that the CSA risk measure is more sensitive to the current information than the true

VaR and the GA VaR. Moreover, the relative importance of the GA correction w.r.t. the CSA VaR is more pronounced for small values of α, that is, for less extreme risks. To summarize, Figure 3 shows that the CSA VaR can either underestimate or overestimate the risk, the GA for filtering can dominate the GA for risk, and the complete GA can yield a good approximation of the true VaR even for portfolio sizes of some hundreds of contracts, at least in the specific linear RFM considered in this illustration.

6 Stochastic Probability of Default and Expected Loss Given Default

A careful analysis of default risk has to consider jointly the default indicator and the loss given default (i.e. one minus the recovery rate). The joint dynamics of the associated dated probability of default and expected loss given default have been studied in a limited number of papers. A well-known stylized fact is the positive correlation between probability of default and loss given default [see e.g. Altman, Brady, Resti, Sironi (2005)]. However, this correlation is a crude summary statistic of the link between the two variables. This link is better understood by introducing time-varying determinants. Observable determinants considered in the literature include the business cycle, the GDP growth rate [see Bruche, Gonzalez-Agrado (2010)], but also the rate of unemployment [Grunert, Weber (2009)]. In fact there exist arguments for a negative link in some circumstances. For instance, the bank has the possibility to declare the default of a borrower when such a default is expected, even if the interest on the debt continues to be regularly paid by the borrower. A too prudent bank can declare defaulted a borrower able to pay the remaining balance, and then create artificially a kind of prepayment. In such a case the probability of default increases and the loss given default decreases, which implies a negative link between the two risk variables.

To capture such complicated effects and their dynamics in the required capital, it is necessary to consider a model with at least two factors. As in the previous sections, these two factors are assumed unobservable, since the uncertainty in their future evolution has to be taken into account in the reserve amount.

6.1 Two-factor dynamic model

Let us consider a portfolio invested in zero-coupon corporate bonds with a same time-to-maturity and identical exposure at default. The loss on the zero-coupon corporate bond maturing at $t+1$ is:

$$y_{i,t+1} = LGD_{i,t+1} Z_{i,t+1},$$

where $Z_{i,t+1}$ is the default indicator and $LGD_{i,t+1}$ is the loss given default. Conditional on the path of the bivariate factor $F_t = (F_{1,t}, F_{2,t})'$, the default indicator $Z_{i,t+1}$ and the loss given default $LGD_{i,t+1}$ are independent, such that $Z_{i,t+1} \sim \mathcal{B}(1, F_{1,t+1})$ and $LGD_{i,t+1}$ admits a beta distribution Beta(a_{t+1}, b_{t+1}) with conditional mean and volatility given by:

$$E\left[LGD_{i,t+1}|F_{t+1}\right] = F_{2,t+1}, \quad V\left[LGD_{i,t+1}|F_{t+1}\right] = \gamma F_{2,t+1}(1 - F_{2,t+1}),$$

where the concentration parameter $\gamma \in (0,1)$ is constant. The parameters of the beta conditional distribution of $LGD_{i,t+1}$ are $a_{t+1} = (1/\gamma - 1)\, F_{2,t+1}$ and $b_{t+1} = (1/\gamma - 1)\,(1 - F_{2,t+1})$. The concentration parameter γ measures the variability of the conditional distribution of $LGD_{i,t+1}$ given F_{t+1} taking into account that the variance of a random variable on $[0,1]$ with mean μ, say, is upper bounded by $\mu\,(1-\mu)$. [10] When the conditional concentration parameter γ approaches 0, the beta distribution degenerates to a point mass; when the conditional concentration parameter γ approaches 1 the beta distribution converges to a Bernoulli distribution. The dynamic factors $F_{1,t}$ and $F_{2,t}$ correspond to the conditional Probability of Default and the conditional Expected Loss Given Default, respectively [11]. The effect of factor F_2 on the beta distribution of expected loss given default is illustrated in Figure 4.

[Insert Figure 4: Conditional distribution of $LGD_{i,t}$ given F_t.]

The factor impacts both the location and shape of the distribution.

Both stochastic factors $F_{1,t}$ and $F_{2,t}$ admit values in the interval $(0,1)$. We assume that the transformed factors $F_t^* = (F_{1,t}^*, F_{2,t}^*)'$ defined by $F_{l,t}^* = \log[F_{l,t}/(1 - F_{l,t})]$, for $l = 1,2$ (logistic

[10] This shows that the mean and the variance cannot be fixed independently for the distribution of a random variable on $[0,1]$.

[11] In the standard credit risk models, the LGD is often assumed constant. In such a case the LGD coincides with both its conditional and unconditional expectations. In our framework the LGD is stochastic as well as its conditional expectation.

transformation), follow a bivariate Gaussian VAR(1) process:

$$F_t^* = c + \Phi F_{t-1}^* + \varepsilon_t,$$

with $\varepsilon_t \sim IIN(0, \Omega)$ and $\Omega = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. The parameters of the factor dynamics are given in Table 1.

Table 1: Parameters of the factor dynamics

	c_1	c_2	Φ_{11}	Φ_{12}	Φ_{21}	Φ_{22}	σ_1	σ_2	ρ	γ
S_1	-1.517	-0.190	0.5	0	0	0.5	0.386	0.655	0.5	0.10
S_2	-1.517	-0.032	0.5	0	0	0.5	0.386	0.661	-0.5	0.10

We consider two parameter sets S_1 and S_2, that correspond to different values of the correlation ρ between shocks in the two transformed factors, namely 0.5 and -0.5, respectively. Thus, we cover the cases of positive (resp. negative) dependence between default and loss given default factors. For both parameter sets, the matrix Φ of the autoregressive coefficients of the transformed factors is such that the individual series feature a first-order autocorrelation coefficient equal to $\Phi_{11} = \Phi_{22} = 0.5$, while the cross effects from the lagged values are $\Phi_{12} = \Phi_{21} = 0$. Moreover, parameters c_1 and σ_1 are such that the unconditional probability of default $PD = E[Z_{i,t}] = E[F_{1,t}]$ and default correlation:

$$\rho_D = corr(Z_{i,t}, Z_{j,t}) = \frac{V[F_{1,t}]}{E[F_{1,t}](1 - E[F_{1,t}])}, \quad i \neq j,$$

are equal to $PD = 0.05$ and $\rho_D = 0.01$, respectively. The PD value corresponds to a rating B [see e.g. Gourieroux, Jasiak (2010)] and the ρ_D value is compatible with the Basle formula for asset correlation and rating B. Both PD and ρ_D involve the marginal distribution of the default factor $F_{1,t}$ only, and thus the values of parameters c_1 and σ_1 are the same across different choices of shock correlation ρ. The concentration parameter in the conditional distribution of loss given default is $\gamma = 0.10$. Finally, to fix the remaining two parameters c_2 and σ_2, we use that the unconditional expected loss given default is (see Appendix 5):

$$ELGD = E[LGD_{i,t} | Z_{i,t} = 1] = E^*[F_{2,t}],$$

and the unconditional variance of the loss given default is:

$$VLGD = V[LGD_{i,t}|Z_{i,t}=1] = \gamma ELGD(1-ELGD) + (1-\gamma)V^*[F_{2,t}],$$

where $E^*[\cdot]$ and $V^*[\cdot]$ denote expectation and variance w.r.t. the new probability measure defined by $E^*[W] = E[WF_{1,t}]/E[F_{1,t}] = E[W|Z_{i,t}=1]$, for any random variable W. For both values of shock correlation ρ, we set c_2 and σ_2 such that $ELGD = 0.45$ and $VLGD = 0.05$. Parameter c_2 is negatively related to ρ. Indeed, the larger the correlation between default and loss given default factors, the smaller the unconditional mean of the loss given default factor that guarantees the same level of ELGD. Parameter σ_2 is slightly decreasing w.r.t. shock correlation ρ.

6.2 CSA VaR and Granularity Adjustment

The CSA VaR and the GA are derived from the results in Section 5.2. Let us first consider the cross-sectional factor approximations. It is proved in Appendix 5 that:

$$\hat{f}_{1,n,t} = N_t/n, \tag{6.1}$$

where $N_t = \sum_{i=1}^{n} \mathbb{1}_{y_{i,t}>0}$ is the number of defaults at date t, and:

$$\begin{aligned}\hat{f}_{2,n,t} \quad = \quad & \arg\max_{f_{2,t}} \Bigg\{ f_{2,t}\left(\frac{1-\gamma}{\gamma}\right)\sum_{i:y_{i,t}>0}\log\left(\frac{y_{i,t}}{1-y_{i,t}}\right) \\ & -N_t\log\Gamma\left[\left(\frac{1-\gamma}{\gamma}\right)f_{2,t}\right] - N_t\log\Gamma\left[\left(\frac{1-\gamma}{\gamma}\right)(1-f_{2,t})\right]\Bigg\},\end{aligned} \tag{6.2}$$

where the sum is over the companies that default at date t, and $\Gamma(.)$ denotes the Gamma function. Thus, the approximation $\hat{f}_{1,n,t}$ of the conditional PD is the cross-sectional default frequency at date t, while the approximation $\hat{f}_{2,n,t}$ of the conditional ELGD is obtained by maximizing the cross-sectional likelihood associated with the conditional beta distribution of the LGD at date t. Proposition 2 on the approximate filtering distribution can be easily extended to multiple factor. Since the cross-sectional log-likelihood can be written as the sum of a component involving $f_{1,t}$ only and a component involving $f_{2,t}$ only, the approximate filtering distribution is such that $F_{1,t}$ and $F_{2,t}$ are independent conditional on information $I_{n,t}$ at order $1/n$, with Gaussian distributions

$N\left(\hat{f}_{l,n,t}+\frac{1}{n}\mu_{l,n,t},\frac{1}{n}J^{-1}_{l,n,t}\right)$, for $l=1,2$, where:

$$\begin{aligned}\mu_{1,n,t} &= -e_1'\Omega^{-1}(\hat{f}^*_{n,t}-c-\Phi\hat{f}^*_{n,t-1}),\\ \mu_{2,n,t} &= -\frac{J^{-1}_{2,n,t}}{\hat{f}_{2,n,t}(1-\hat{f}_{2,n,t})}\left[e_2'\Omega^{-1}(\hat{f}^*_{n,t}-c-\Phi\hat{f}^*_{n,t-1})+1-2\hat{f}_{2,n,t}\right]+\frac{1}{2}J^{-2}_{2,n,t}K_{2,n,t},\end{aligned} \tag{6.3}$$

with $\hat{f}^*_{n,t}=\left(\log[\hat{f}_{1,n,t}/(1-\hat{f}_{1,n,t})],\log[\hat{f}_{2,n,t}/(1-\hat{f}_{2,n,t})]\right)'$ and vectors $e_1=(1,0)'$, $e_2=(0,1)'$, and:

$$\begin{aligned}J_{1,n,t} &= \frac{1}{\hat{f}_{1,n,t}(1-\hat{f}_{1,n,t})},\\ J_{2,n,t} &= \hat{f}_{1,n,t}\left(\frac{1-\gamma}{\gamma}\right)^2\left\{\Psi'\left[\left(\frac{1-\gamma}{\gamma}\right)\hat{f}_{2,n,t}\right]+\Psi'\left[\left(\frac{1-\gamma}{\gamma}\right)(1-\hat{f}_{2,n,t})\right]\right\},\end{aligned} \tag{6.4}$$

with $\Psi(s)=\frac{d\log\Gamma(s)}{ds}$ and:

$$K_{2,n,t} = -\hat{f}_{1,n,t}\left(\frac{1-\gamma}{\gamma}\right)^3\left\{\Psi''\left[\left(\frac{1-\gamma}{\gamma}\right)\hat{f}_{2,n,t}\right]-\Psi''\left[\left(\frac{1-\gamma}{\gamma}\right)(1-\hat{f}_{2,n,t})\right]\right\}. \tag{6.5}$$

Let us now derive the CSA VaR and the GA. From the results in Section 5.2 iv), we get the next Proposition.

Proposition 5: *In the model with stochastic conditional PD and ELGD:*

i) *The CSA VaR at risk level α is the solution of the equation:*

$$a\left(VaR_\infty(\alpha;\hat{f}_{n,t}),0;\hat{f}_{n,t}\right)=\alpha,$$

where:

$$\begin{aligned}a(w,0;f_t) &= \Phi\left(\frac{\log[w/(1-w)]-c_{1,t}}{\sigma_1}\right)\\ &\quad+\int_w^1\Phi\left[\frac{\log[w/(y-w)]-c_{2,t}-\frac{\rho\sigma_2}{\sigma_1}(\log[y/(1-y)]-c_{1,t})}{\sigma_2\sqrt{1-\rho^2}}\right]\\ &\quad\cdot\frac{1}{\sigma_1}\varphi\left(\frac{\log[y/(1-y)]-c_{1,t}}{\sigma_1}\right)\frac{1}{y(1-y)}dy.\end{aligned}$$

and $c_{l,t}=c_l+\Phi_{l,1}\log[f_{1,t}/(1-f_{1,t})]+\Phi_{l,2}\log[f_{2,t}/(1-f_{2,t})]$*, for* $l=1,2$.

ii) *The GA at risk level* α *is* $GA_{n,t}(\alpha) = GA_{risk,t}(\alpha) + GA_{filt,t}(\alpha)$*, where* $GA_{risk,t}(\alpha)$ *is computed from Proposition 3 with* $g_\infty(w; f_t) = b(w, 0; f_t)$ *and:*

$$E[\sigma^2(F_{t+1})|m(F_{t+1}) = w, F_t = f_t] = w(\gamma - w) + (1-\gamma)w\frac{b(w,1;f_t)}{b(w,0;f_t)},$$

where:

$$b(w,k;f_t) = \int_w^1 \frac{y^k}{\sigma_1\sigma_2}\varphi\left(\frac{\log[w/(y-w)] - c_{1,t}}{\sigma_1}, \frac{\log[y/(1-y)] - c_{2,t}}{\sigma_2}; \rho\right)\frac{1}{(1-y)w(y-w)}dy,$$

and $\varphi(.,.;\rho)$ *denotes the pdf of the standard bivariate Gaussian distribution with correlation* ρ*; the component* $GA_{filt,t}(\alpha)$ *is given by:*

$$\begin{aligned} GA_{filt,t}(\alpha) \quad = \quad & -\frac{1}{g_\infty[VaR_\infty(\alpha;\hat{f}_{n,t});\hat{f}_{n,t}]}\sum_{l=1}^{2}\left\{\frac{\partial a[VaR_\infty(\alpha,\hat{f}_{nt}),0;\hat{f}_{l,nt}]}{\partial f_{l,t}}\mu_{l,nt}\right. \\ & \left. + \frac{1}{2}J^{-1}_{l,n,t}\frac{\partial^2 a[VaR_\infty(\alpha;\hat{f}_{nt}),0;\hat{f}_{l,nt}]}{\partial f^2_{l,t}}\right\}, \end{aligned}$$

where $\mu_{l,n,t}$ *and* $J_{l,n,t}$ *are given in (6.3) and (6.4).*

Proof: See Appendix 5.

The CSA and GA VaR in Proposition 5 are given in closed form, up to a few one-dimensional integrals. We provide the CSA VaR, GA VaR and its risk and filtering components in Figures 5 and 6, for portfolio size $n = 500$ and risk level $\alpha = 99.5\%$.

[Insert Figure 5: CSA VaR and GA VaR as functions of the factor values, parameter set S_1.]

[Insert Figure 6: CSA VaR and GA VaR as functions of the factor values, parameter set S_2.]

As expected, the required capital, measured by either CSA VaR, or GA VaR, is increasing with respect to both factors F_1 and F_2, with larger sensitivity to factor F_1. The risk GA component (before adjusting for the portfolio size) is always positive, more sensitive to factor F_2 and its range is much smaller than the range of the filtering GA component. The filtering component can take both positive and negative values and depends nonlinearly on factor F_1 in a decreasing way. This is a consequence of the properties of the ML estimator for the parameter of a Bernoulli distribution with parameter F_1. Indeed, it is known that the estimator is very accurate when F_1 is close to 0 (or

1). By comparing Figures 5 and 6, we observe immediately that the CSA VaR and the GA VaR are smaller when the two risks are negatively correlated. We can observe a change in the sign of dependence of the filtering component with respect to the second factor.

Let us now consider the dynamics of the risk factors and risk measures.

[Insert Figure 7: Time series of factor values and approximations, portfolio losses and default frequencies.]

[Insert Figure 8: Time series of CSA VaR, GA VaR, GA risk and filtering components.]

The two first panels in Figure 7 provide the factors and their cross-sectional approximations corresponding to a simulated path of factors and individual risks, with parameter set S_1 and portfolio size $n = 500$. The cross-sectional approximations are accurate for both factors, and especially so for the second factor which is driving the quantitative risk. The third panel provides the series of percentage portfolio losses and default frequencies. The default frequency is a percentage portfolio loss with zero recovery rate, which explains why this series is systematically larger. The default frequency is driven by factor F_1 only, and the non parallel evolution of the two series is due to factor F_2, including its dependence with F_1. The upper panel of Figure 8 provides the time series of CSA VaR and GA VaR. The granularity adjustment is in general positive and rather small, except at dates at which the CSA VaR is low. At these dates the adjustment is mainly due to its filtering component, as can be deduced from the lower panel of Figure 8. The risk component is almost constant over time, and (much) smaller in absolute value than the filtering component at most dates.

The capital adequacy is usually checked by performing some backtesting. It is known that the true conditional VaR is such that:

$$E[H_t | I_{n,t-1}] = 0, \tag{6.6}$$

where $H_t = \mathbb{1}_{W_{n,t}/n \geq VaR_{n,t-1}(\alpha)} - (1 - \alpha)$ [see equation (5.2)]. This is a conditional moment restriction, which can be used to construct a battery of specification tests [Giacomini, White (2006)]. More precisely, let us consider an instrument X_{t-1}, that is a function of information $I_{n,t-1}$, and deduce from (6.6) the unconditional moment restriction $E[H_t X_{t-1}] = 0$. When $X_{t-1} = 1$, we get the simple condition $E[H_t] = 0$, which is the basis for the standard backtesting procedure relying on the number of violations and suggested by the regulator. More secure backtesting is obtained by considering different instruments. We rewrite the moment condition $E[H_t X_{t-1}] = 0$ in terms of correlation as $corr(H_t, X_{t-1}) = 0$, and provide in Table 2 the true values of the backtesting moments and correlations for the portfolio sizes $n = 250$ and $n = 500$, and the CSA VaR and GA VaR.

[Insert Table 2: Backtesting of CSA VaR and GA VaR.]

Let us for instance consider the second row of the upper panel (resp. the second row of the lower panel). With the CSA VaR suggested in the standard regulation the probability of violation is 1.2% for $n = 250$ (resp. 0.8% for $n = 500$) instead of 0.5%. When the granularity adjustment is applied this probability becomes 0.4% (resp. 0.5%). This shows clearly that the reserve computed with the CSA approximation is too low on average. The better accuracy of the GA VaR is confirmed when we consider other instruments such as the lagged factor approximations, or lagged values of H. We observe that the associated correlations are typically closer to 0 after granularity adjustment.

Finally, it is interesting to discuss the possible effect of the number of factors.

[Insert Figure 9: CSA VaR and GA VaR as functions of the correlation parameter in a two-factor model.]

In Figure 9 we plot the CSA VaR, the GA VaR accounting for risk only, and the GA VaR with full correction as functions of the correlation parameter ρ, for $\rho \in (-1, 1)$ and portfolio size $n = 500$. The limiting values $\rho = \pm 1$ correspond to a one-factor model, with the single factor impacting both the probability of default and the loss given default. We observe that the granularity adjustment for risk is rather small, and close to zero when the correlation parameter is close to the limiting values. At the contrary, the adjustment for filtering becomes larger when the absolute value of the correlation parameter increases. As a result, the total GA is almost independent of ρ when $\rho \in (-0.6, 0.6)$, but the relative contributions of the risk and filtering components varies with ρ. The total GA explodes when ρ approaches the limiting values ± 1. This singularity reflects the discountinuity in the number of factors.

7 Concluding Remarks

For large homogenous portfolios and a variety of both single-factor and multi-factor dynamic risk models, closed form expressions of the VaR and other distortion risk measures can be derived at order $1/n$. Two granularity adjustments are required. The first GA concerns the conditional VaR with current factor value assumed to be observed. The second GA takes into account the unobservability of the current factor value and is specific to dynamic factor models. This explains why this GA has not been taken into account in the earlier literature which focuses on static models.

These GA assume given the function linking the individual risks to factors and idiosyncratic risks, and also the distributions of both the factor and idiosyncratic risks. In practice the link function and the distribution depend on unknown parameters, which have to be estimated. This creates an additional error on the VaR, which has been considered neither here, nor in the previous literature. This estimation error can be larger than the GA derived in this paper. However, such a separate analysis is compatible with the Basel 2 methodology. Indeed, the GA in this paper are useful to compute the reserves for **Credit Risk**, whereas the adjustment for estimation concerns the reserves for **Estimation Risk**.

The granularity adjustment principle appeared in Pillar 1 of the New Basel Capital Accord in 2001 [BCBS (2001)], concerning the minimum capital requirement. It has been suppressed from Pillar 1 in the most recent version of the Accord in 2003 [BCBS (2003)], and assigned to Pillar 2 on internal risk models. The recent financial crisis has shown that systematic risks, which include in particular systemic risks [12], have to be distinguished from unsystematic risk, and in the new organization these two risks will be supervised by different regulators. This shows the importance of taking into account this distinction in computing the reserves, that is, also at Pillar 1 level. For instance, one may fix different risk levels α_1 and α_2 in the CSA and GA VaR components, and smooth differently these components over the cycle in the definition of the required capital. The recent literature on granularity shows that the technology is now in place and can be implemented not only for static linear factor models, but also for nonlinear dynamic factor models.

[12] A systemic risk is a systematic risk which can seriously damage the Financial System.

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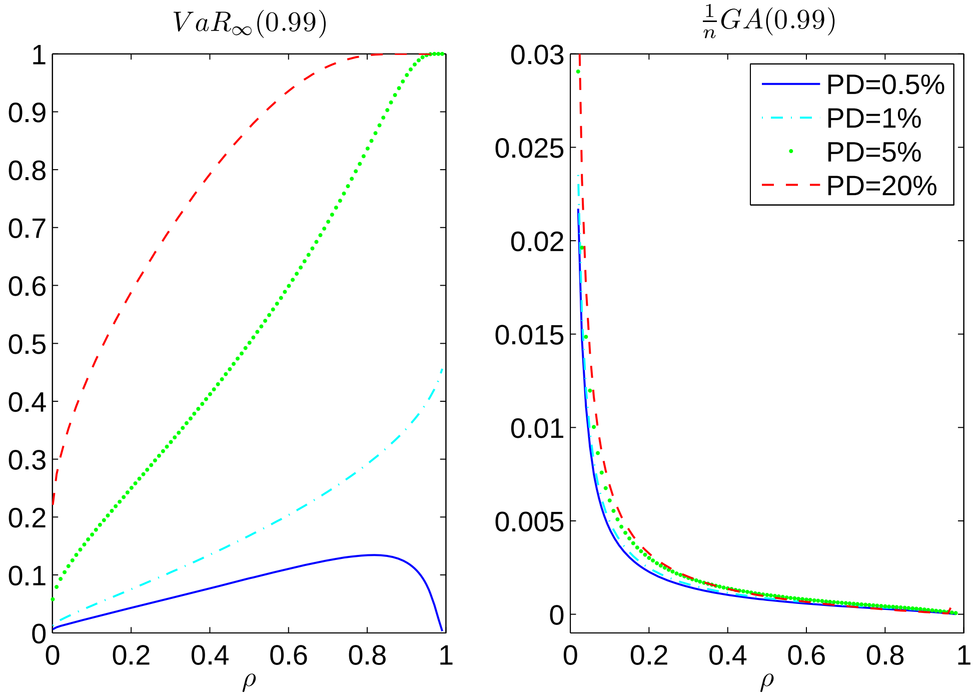
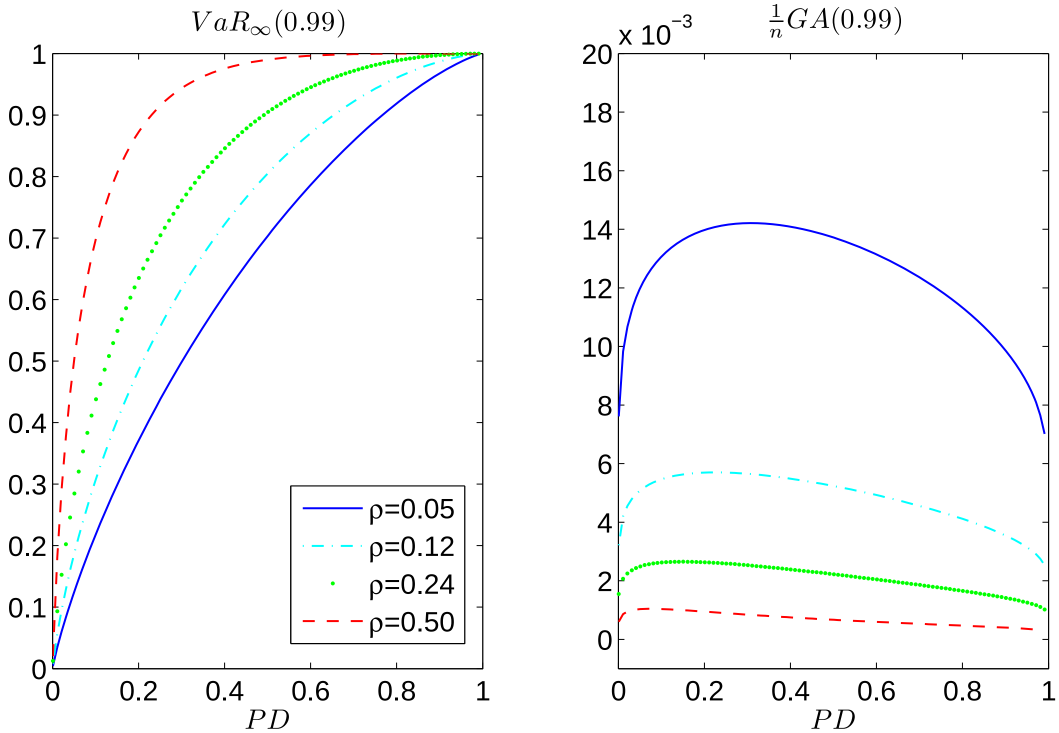
Figure 1: CSA CreditVaR and granularity adjustment as functions of the asset correlation in the Merton-Vasicek model.



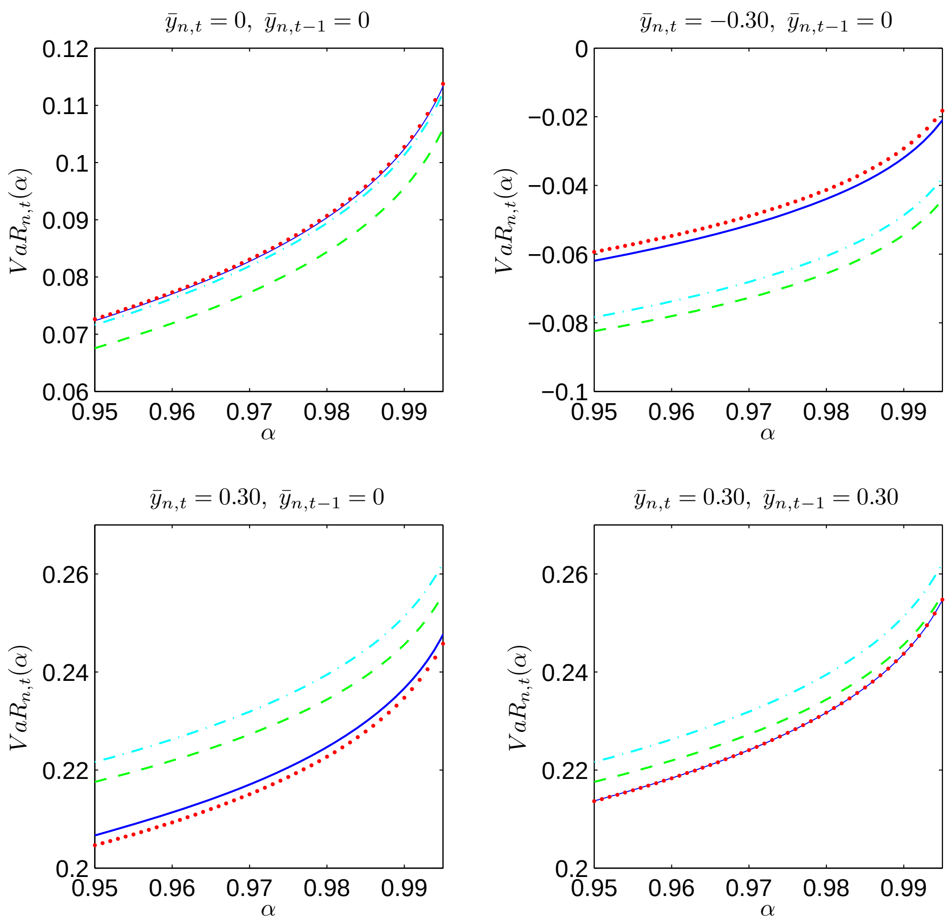
The panels display the CSA quantile $VaR_\infty(\alpha)$ (left) and the granularity adjustment per contract $\frac{1}{n}GA(\alpha)$ (right) as functions of the asset correlation ρ. The confidence level is $\alpha = 0.99$ and the portfolio size is $n = 1000$. In each panel, the curves correspond to different values of the unconditional probability of default, that are $PD = 0.5\%$ (solid line), $PD = 1\%$ (dashed dotted line), $PD = 5\%$ (dotted line), and $PD = 20\%$ (dashed line).

Figure 2: CSA CreditVaR and granularity adjustment as a function of the unconditional probability of default in the Merton-Vasicek model.



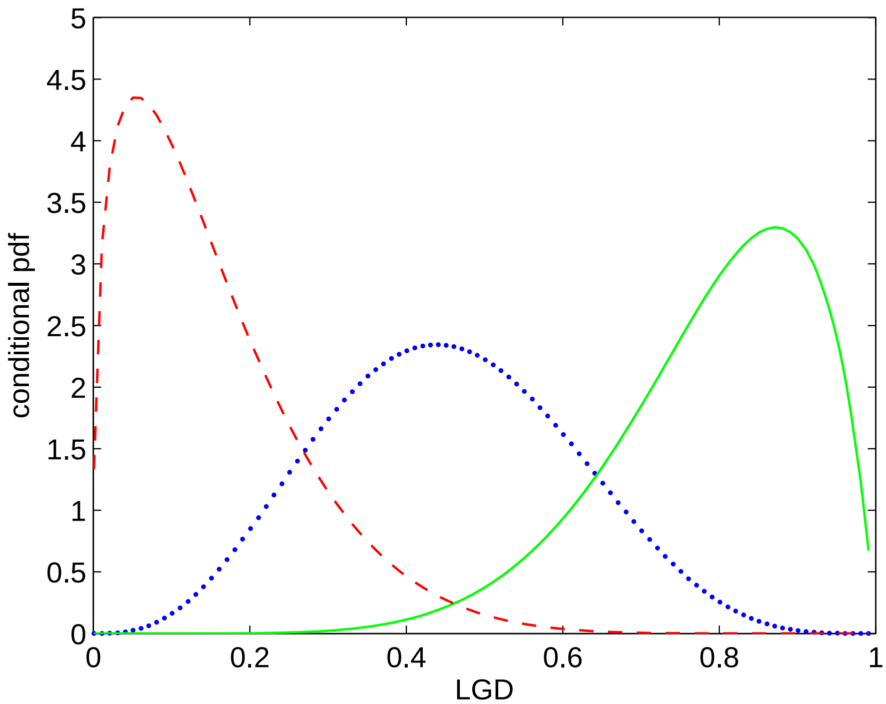
The panels display the CSA quantile $VaR_\infty(\alpha)$ (left) and the granularity adjustment per contract $\frac{1}{n}GA(\alpha)$ (right) as functions of the unconditional probability of default PD. The confidence level is $\alpha = 0.99$ and the portfolio size is $n = 1000$. In each panel, the curves correspond to different values of the asset correlation, that are $\rho = 0.05$ (solid line), $\rho = 0.12$ (dashed dotted line), $\rho = 0.24$ (dotted line), and $\rho = 0.50$ (dashed line).

Figure 3: VaR as a function of the risk level in the linear RFM with AR(1) factor.



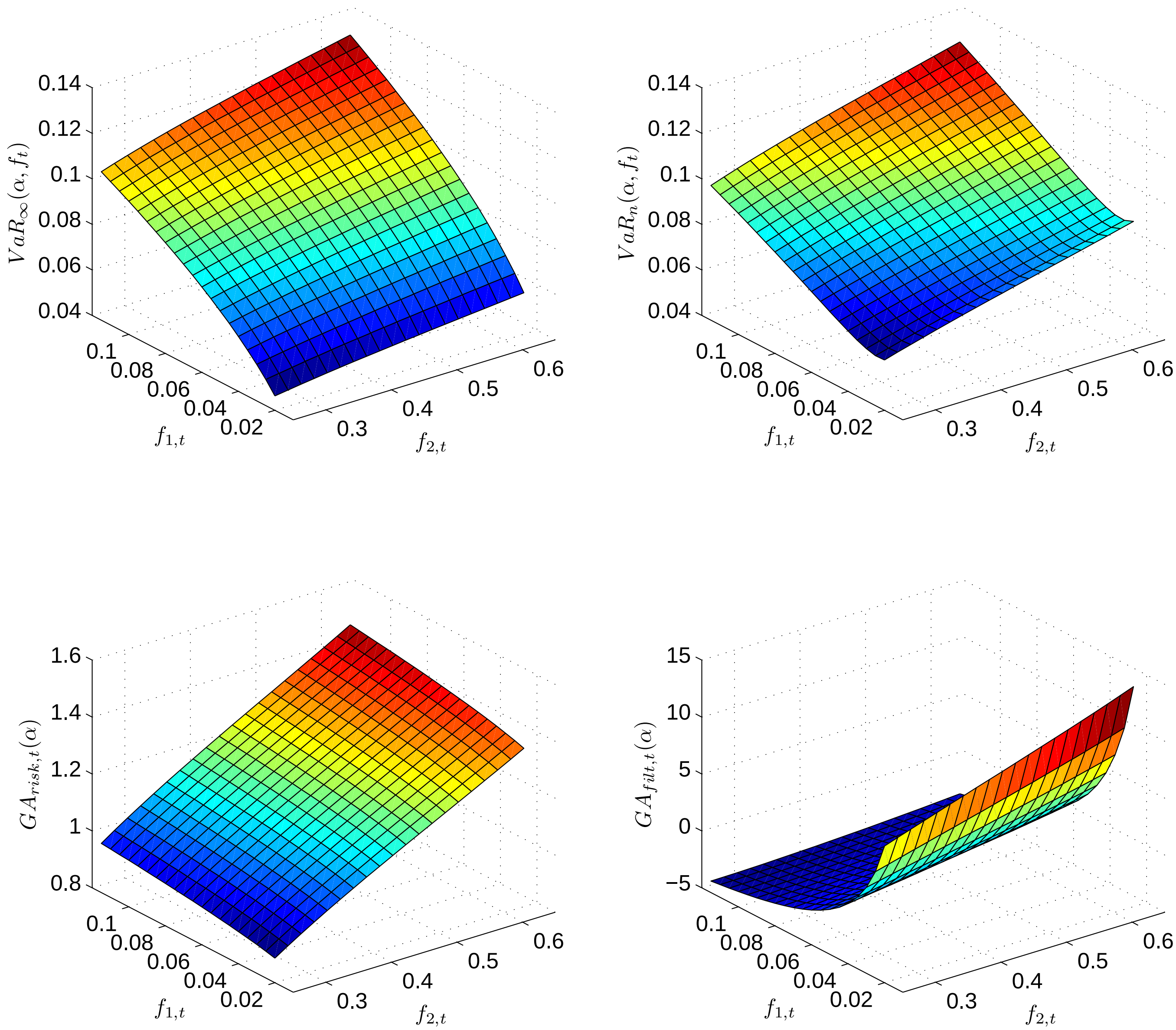
In each Panel we display the true VaR (solid line), the CSA VaR (dashed line), the GA VaR accounting for risk only (dashed-dotted line) and the GA VaR accounting for both risk and filtering (dotted line), as a function of the confidence level α. The four Panels correspond to different available information $I_{n,t}$, that are $\bar{y}_{n,t} = \bar{y}_{n,t-1} = 0$ in the upper left Panel, $\bar{y}_{n,t} = -0.30$, $\bar{y}_{n,t-1} = 0$ in the upper right Panel, $\bar{y}_{n,t} = 0.30$, $\bar{y}_{n,t-1} = 0$ in the lower left Panel, and $\bar{y}_{n,t} = 0.30$, $\bar{y}_{n,t-1} = 0.30$ in the lower right Panel, respectively. The portfolio size is $n = 100$. The model parameters are such that the unconditional standard deviation of the individual risks is 0.15, the unconditional correlation between individual risks is 0.10, the factor mean is $\mu = 0$, and the factor autoregressive coefficient is $\rho = 0.5$.

Figure 4: Conditional distribution of $LGD_{i,t}$ given F_t.



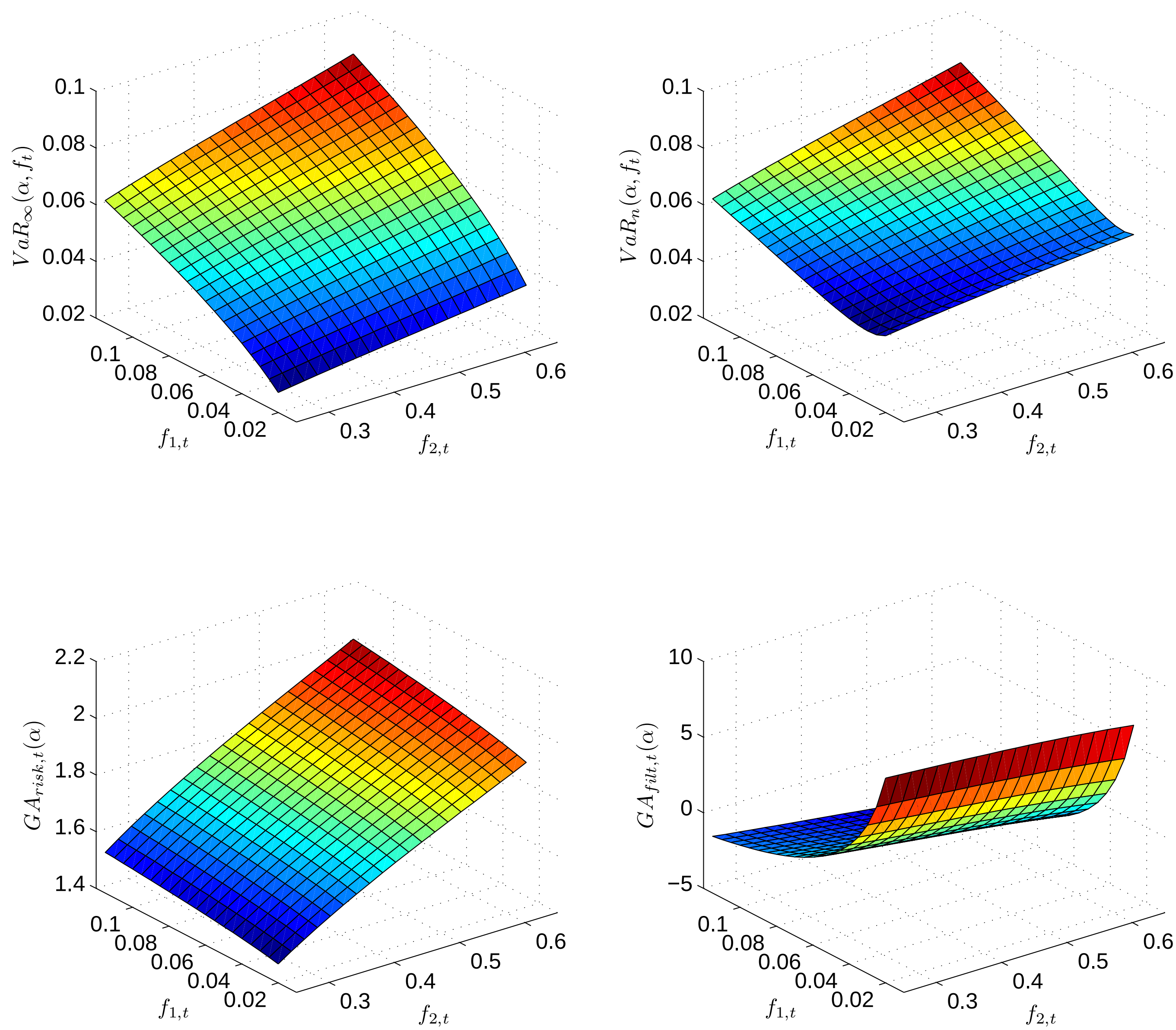
The figure displays the conditional pdf of $LGD_{i,t}$ given F_t for different values of $F_{2,t} = e^{F^*_{2,t}}/(1+e^{F^*_{2,t}})$. The values of the transformed factor $F^*_{2,t}$ are $F^*_{2,t} = \mu_2$ (dotted line), $F^*_{2,t} = \mu_2 - 2\nu_2$ (dashed line) and $F^*_{2,t} = \mu_2 + 2\nu_2$ (solid line), where $\mu_2 = \dfrac{c_2}{1-\Phi_{22}}$ and $\nu_2^2 = \dfrac{\sigma_2^2}{1-\Phi_{22}^2}$ are the stationary mean and variance of $F^*_{2,t}$ and the parameter values are given in Table 1.

Figure 5: CSA VaR and GA VaR as functions of the factor values, parameter set S_1.



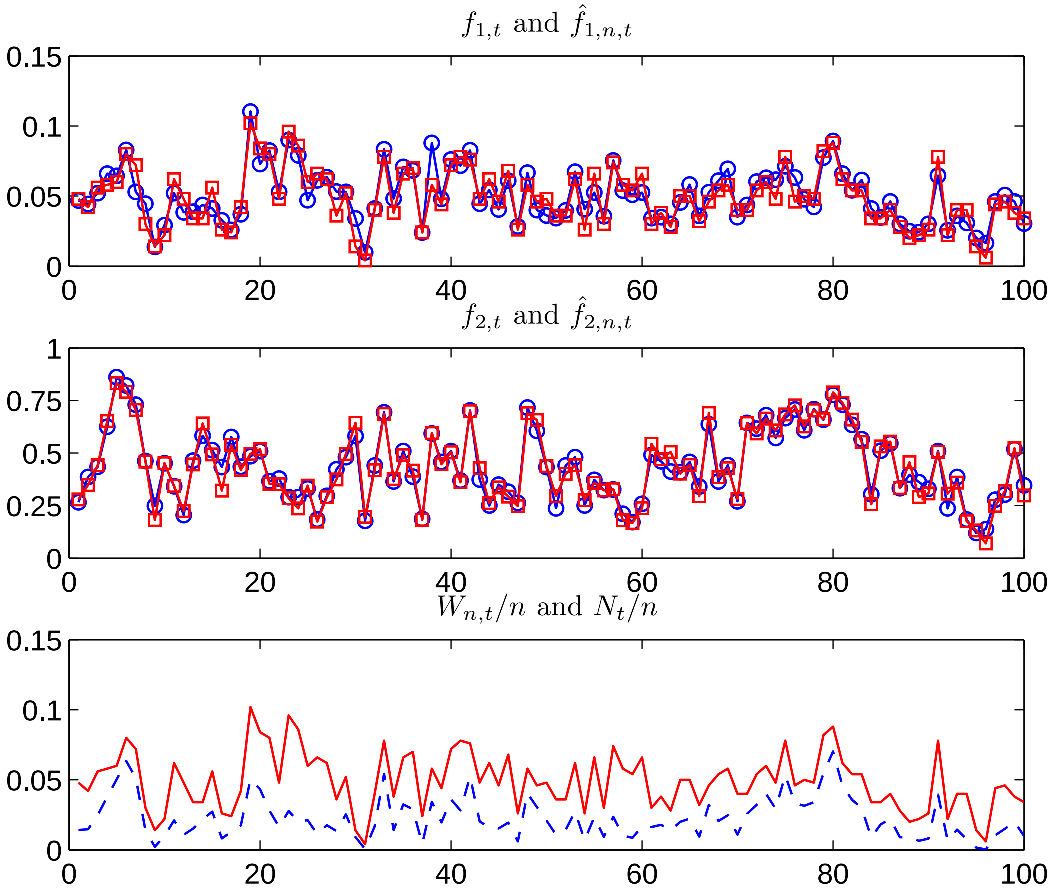
The four panels display the CSA VaR $VaR_\infty(\alpha; f_t)$, the GA VaR $VaR_n(\alpha; f_t)$, the GA for risk $GA_{risk,t}(\alpha)$ and the GA for filtering $GA_{filt,t}(\alpha)$, respectively, as functions of the factor values $f_t = (f_{1,t}, f_{2,t})'$. The lagged factor values are such that $f^*_{t-1} = (Id_2 - \Phi)^{-1}c$ is equal to the stationary mean of the transformed factor process. The risk level is $\alpha = 99.5\%$. The GA VaR is computed for portfolio size $n = 500$. The parameters correspond to set S_1 in Table 1.

Figure 6: CSA VaR and GA VaR as functions of the factor values, parameter set S_2.



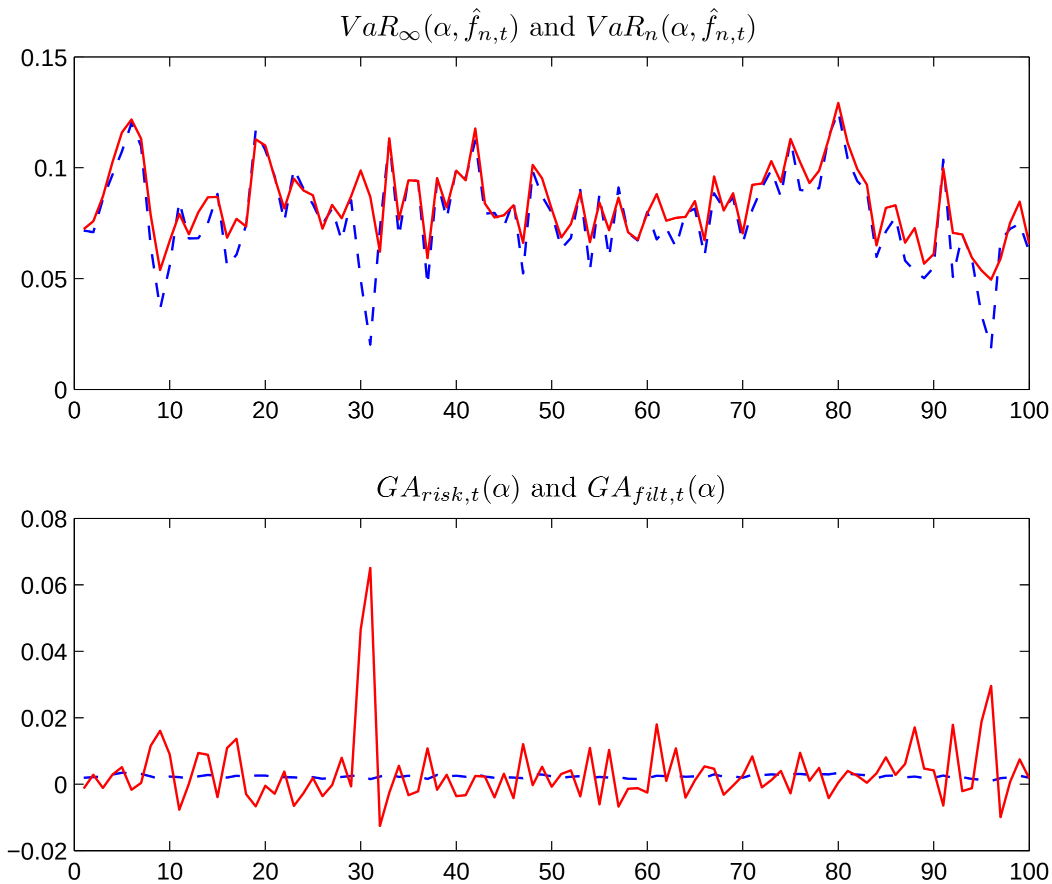
The four panels display the CSA VaR $VaR_\infty(\alpha; f_t)$, the GA VaR $VaR_n(\alpha; f_t)$, the GA for risk $GA_{risk,t}(\alpha)$ and the GA for filtering $GA_{filt,t}(\alpha)$, respectively, as functions of the factor values $f_t = (f_{1,t}, f_{2,t})'$. The lagged factor values are such that $f_{t-1}^* = (Id_2 - \Phi)^{-1}c$ is equal to the stationary mean of the transformed factor process. The risk level is $\alpha = 99.5\%$. The GA VaR is computed for portfolio size $n = 500$. The parameters correspond to set S_2 in Table 1.

Figure 7: Time series of factor values and approximations, portfolio losses and default frequencies



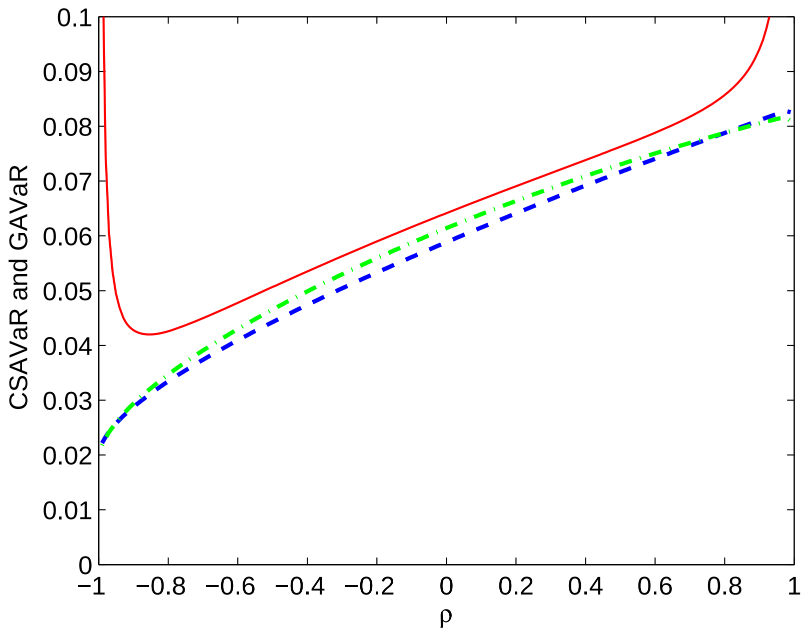
The first two panels display a simulated time series of factor values $f_{1,t}$ and $f_{2,t}$ (circles) and factor approximations $\hat{f}_{1,n,t}$ and $\hat{f}_{2,n,t}$ (squares) for portfolio size $n = 500$. The third panel displays the associated default frequency (solid line) and standardized portfolio losses $W_{n,t}/n$ (dashed line). The parameters correspond to set S_1 in Table 1.

Figure 8: Time series of CSA VaR, GA VaR, GA risk and filtering components.



The upper panel displays the CSA VaR $VaR_\infty(\alpha; \hat{f}_{n,t})$ (dashed line) and the GA VaR $VaR_n(\alpha; \hat{f}_{n,t})$ (solid line). The lower panel display the risk (dashed line) and filtering (solid line) components of the GA. The risk level is $\alpha = 99.5\%$ and the portfolio size is $n = 500$. The parameters correspond to set S_1 in Table 1.

Figure 9: CSA and GA VaR as functions of the correlation parameter in a two-factor model.



The figure displays the CSA VaR (dashed line), the GA VaR accounting for risk only (dashed-dotted line) and the GA VaR accounting for both risk and filtering (solid line) as functions of the correlation parameter ρ in a two-factor model. The autoregressive parameters are $\Phi_{11} = \Phi_{22} = 0.5$, $\Phi_{12} = \Phi_{21} = 0$, parameters c_1 and σ_1 are such that $PD = 5\%$ and $\rho_D = 1\%$, and parameter γ is equal to $\gamma = 0.10$, as in Table 1. For each value of ρ, parameters c_2 and σ_2 are such that $ELGD = 0.45$ and $VLGD = 0.05$. The current factor approximations are $\hat{f}_{1,n,t} = 0.04$ and $\hat{f}_{2,n,t} = 0.40$, the lagged transformed factor approximations $\hat{f}^*_{1,n,t-1}$ and $\hat{f}^*_{2,n,t-1}$ are equal to the unconditional means of the transformed factors. The portfolio size is $n = 500$.

Table 2: Backtesting of CSA VaR and GA VaR.

$n = 250$	CSA	GA
$E[H_t]$	0.007	-0.001
$Corr(H_t, H_{t-1})$	-0.008	0.000
$Corr(H_t, H_{t-2})$	-0.002	0.000
$Corr\left(H_t, \hat{f}_{1,n,t-1}\right)$	-0.065	0.023
$Corr\left(H_t, \hat{f}_{1,n,t-2}\right)$	-0.008	0.019
$Corr\left(H_t, \hat{f}_{2,n,t-1}\right)$	-0.001	0.000
$Corr\left(H_t, \hat{f}_{2,n,t-2}\right)$	-0.001	0.001
$n = 500$	CSA	GA
$E[H_t]$	0.003	-0.000
$Corr(H_t, H_{t-1})$	-0.003	0.001
$Corr(H_t, H_{t-2})$	-0.003	-0.003
$Corr\left(H_t, \hat{f}_{1,n,t-1}\right)$	-0.027	0.012
$Corr\left(H_t, \hat{f}_{1,n,t-2}\right)$	-0.005	0.010
$Corr\left(H_t, \hat{f}_{2,n,t-1}\right)$	-0.004	-0.001
$Corr\left(H_t, \hat{f}_{2,n,t-2}\right)$	-0.002	-0.001

The variable $H_t = \mathbf{1}_{W_{n,t}/n \geq VaR_{n,t-1}(\alpha)} - (1-\alpha)$ is computed by using $VaR_{n,t-1}(\alpha) = VaR_\infty(\alpha; \hat{f}_{n,t-1})$ for the CSA VaR and $VaR_{n,t-1}(\alpha) = VaR_\infty(\alpha; \hat{f}_{n,t-1}) + \frac{1}{n}[GA_{risk,t-1}(\alpha) + GA_{filt,t-1}(\alpha)]$ for the GA VaR. The confidence level is $\alpha = 0.995$. The parameters correspond to set S_1 in Table 1. All quantities are computed by Monte-Carlo simulation on a time series of length $T = 100000$.

i) Expansion of the cumulative distribution function

Let us consider a pair (X, Y) of real random variables, where X is a continuous random variable with pdf f_1 and cdf G_1. The aim of this section is to derive an expansion of the function:

$$a(x, \varepsilon) = P[X + \varepsilon Y < x], \tag{a.1}$$

in a neighbourhood of $\varepsilon = 0$.

The following Lemma has been first derived by Gouriéroux, Laurent, Scaillet (2000) for the second-order expansion, and by Martin, Wilde (2002) for any order. We will extend the proof in Gouriéroux, Laurent, Scaillet (2000), which avoids the use of characteristic functions and shows more clearly the needed regularity conditions (RC) (see below).

Lemma a.1: *Under regularity conditions (RC), we have:*

$$a(x, \varepsilon) = G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^j}{j!} \varepsilon^j \frac{d^{j-1}}{dx^{j-1}} [g_1(x) E(Y^j | X = x)] \right\} + o(\varepsilon^J).$$

Proof: The proof requires two steps. First, we consider the case of a bivariate continuous random vector (X, Y). Then, we extend the result when Y and X are in a deterministic relationship.

a) Bivariate continuous vector

Let us denote by $g_{1,2}(x, y)$ [resp. $g_{1|2}(x|y)$ and $G_{1|2}(x|y)$] the joint pdf of (X, Y) (resp. the conditional pdf and cdf of X given Y). We have:

$$\begin{aligned}
a(x, \varepsilon) &= P[X + \varepsilon Y < x] = EP[X < x - \varepsilon Y | Y] = E[G_{1|2}(x - \varepsilon Y | Y)] \\
&= E[G_{1|2}(x|Y)] + \sum_{j=1}^{J} \left\{ \frac{(-1)^j \varepsilon^j}{j!} E[Y^j \frac{d^{j-1}}{dx^{j-1}} g_{1|2}(x|Y)] \right\} + o(\varepsilon^J) \\
&= G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^j \varepsilon^j}{j!} \frac{d^{j-1}}{dx^{j-1}} (E[Y^j g_{1|2}(x|Y)]) \right\} + o(\varepsilon^J) \\
&= G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^j \varepsilon^j}{j!} \frac{d^{j-1}}{dx^{j-1}} [g_1(x) E[Y^j | X = x]] \right\} + o(\varepsilon^J).
\end{aligned}$$

b) Variables in deterministic relationship

Let us now consider the case of a function $a(x, \varepsilon) = P[X + \varepsilon c(X) < x]$, where the direction of expansion $Y = c(X)$ is in a deterministic relationship with variable X. Let us introduce a variable Z independent of X with a gamma distribution $\gamma(\nu, \nu)$, and study the function:

$$a(x, \varepsilon; \nu) = P[X + \varepsilon c(X) Z < x].$$

The joint distribution of variables X and $Y^* = c(X)Z$ is continuous; thus, the results of part i) of the proof can be applied. We get:

$$\begin{aligned} a(x,\varepsilon;\nu) &= G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^j \varepsilon^j}{j!} \frac{d^{j-1}}{dx^{j-1}} (g_1(x) E[c(X)^j Z^j | X = x]) \right\} + o(\varepsilon^J) \\ &= G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^j \varepsilon^j}{j!} \frac{d^{j-1}}{dx^{j-1}} [g_1(x) c(x)^j E(Z^j)] \right\} + o(\varepsilon^J) \\ &= G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^j \varepsilon^j}{j!} \mu_j(\nu) \frac{d^{j-1}}{dx^{j-1}} [g_1(x) c(x)^j] \right\} + o(\varepsilon^J), \end{aligned}$$

where:

$$\mu_j(\nu) = E(Z^j) = (1 - \frac{1}{\nu})(1 - \frac{2}{\nu}) \ldots (1 - \frac{j-1}{\nu}), \quad j = 1, \ldots, J.$$

Since the moments $\mu_j(\nu), j = 1, \ldots, J$ tend uniformly to 1, and $Y^* = Zc(X)$ tends to $Y = c(X)$, when ν tends to infinity, we get:

$$\begin{aligned} a(x,\varepsilon) &= P[X + \varepsilon c(X) < x] = \lim_{\nu \to \infty} a(x,\varepsilon;\nu) \\ &= G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^j \varepsilon^j}{j!} \frac{d^{j-1}}{dx^{j-1}} [g_1(x) c(x)^j] \right\} + o(\varepsilon^J). \end{aligned}$$

QED

ii) Application to large portfolio risk

Let us consider the asymptotic expansion (3.4):

$$W_n/n = m(F) + \sigma(F)X/\sqrt{n} + O(1/n),$$

where X is independent of F with distribution $N(0,1)$ and the term $O(1/n)$ is zero-mean, conditional on F. We have:

$$a_n(x) = P[W_n/n < x] = P[m(F) + \sigma(F)X/\sqrt{n} + O(1/n) < x].$$

Then, the expansion in Lemma a.1 can be applied at order 2, noting that:

$$\begin{aligned} &E[O(1/n)|F] = 0, \quad E[\sigma(F)X|m(F)] = E[\sigma(F)|m(F)]E[X] = 0, \\ &E[\sigma^2(F)X^2|m(F)] = E[X^2]E[\sigma^2(F)|m(F)] = E[\sigma^2(F)|m(F)]. \end{aligned}$$

We deduce the following Lemma:

Lemma a.2: *We have:*

$$\begin{aligned} a_n(x) &= P[W_n/n < x] \\ &= P[m(F) < x] + \frac{1}{2n} \frac{d}{dx} \left\{ g_\infty(x) E[\sigma^2(F)|m(F) = x] \right\} + o(1/n), \end{aligned}$$

where g_∞ is the pdf of $m(F)$.

Note that this approximation at order $1/n$ is exact and not itself approximated by the cdf based on a bivariate distribution as in Vasicek (2002).

iii) Expansion of the VaR

The expansion of the VaR per individual is deduced from the Bahadur's expansion [see Bahadur (1966), or Gagliardini, Gouriéroux (2010b), Section 6.2].

Lemma a.3 (Bahadur's expansion): *Let us consider a sequence of cdf's F_n tending to a limiting cdf F_∞ at uniform rate $1/n$:*

$$F_n(x) = F_\infty(x) + O(1/n).$$

Let us denote by Q_n and Q_∞ the associated quantile functions and assume that the limiting distribution is continuous with density f_∞. Then:

$$Q_n(\alpha) - Q_\infty(\alpha) = -\frac{F_n[Q_\infty(\alpha)] - F_\infty[Q_\infty(\alpha)]}{f_\infty[Q_\infty(\alpha)]} + o(1/n).$$

Lemma a.3 can be applied to the standardized portfolio risk. The limiting distribution is the distribution of $m(F)$ with pdf g_∞ and quantile function VaR_∞. By using the expansion of Lemma a.2, we get:

$$\begin{aligned} VaR_n(\alpha) &= VaR_\infty(\alpha) - \frac{1}{2n}\frac{1}{g_\infty[VaR_\infty(\alpha)]}\left[\frac{d}{dx}\left\{g_\infty(x)E[\sigma^2(F)|m(F)=x]\right\}\right]_{x=VaR_\infty(\alpha)} \\ &\quad + o(1/n) \\ &= VaR_\infty(\alpha) - \frac{1}{2n}\left\{\frac{d\log g_\infty}{dx}[VaR_\infty(\alpha)]E[\sigma^2(F)|m(F)=VaR_\infty(\alpha)]\right. \\ &\quad \left. + \left[\frac{d}{dx}E[\sigma^2(F)|m(F)=x]\right]_{x=VaR_\infty(\alpha)}\right\} + o(1/n). \end{aligned}$$

This is the result in Proposition 1.

APPENDIX 2: Aggregation of risk measures

The aggregation of risk measures, in particular of VaR, is a difficult task. In this Appendix we consider this question in a large portfolio perspective for a single factor model. Let us index by $k = 1, \cdots, K$ the different subpopulations, with respective weights π_k, $k = 1, \cdots, K$. Within subpopulation k, the conditional mean and variance given factor F are denoted $m_k(F)$ and $\sigma_k^2(F)$, respectively. At population level, the conditional mean is:

$$\bar{m}(F) = \sum_{k=1}^{K} \pi_k m_k(F), \tag{a.2}$$

by the iterated expectation theorem, and the conditional variance is given by:

$$\bar{\sigma}^2(F) = \sum_{k=1}^{K} \pi_k \sigma_k^2(F) + \sum_{k=1}^{K} \pi_k \left[m_k(F) - \bar{m}(F)\right]^2, \tag{a.3}$$

by the variance decomposition equation. We denote by Q_∞, G_∞ and g_∞ the quantile, cdf and pdf of the distribution of factor F, respectively.

i) Aggregation of CSA risk measures

We assume:

A.3: *The conditional means $m_k(F)$ are increasing functions of F, for any $k = 1, \cdots, K$.*

Under Assumption A.3, the systematic risk components $m_k(F)$, $k = 1, \cdots, K$ are co-monotonic [see McNeil, Frey, Embrechts (2005), Proposition 6.15]. Therefore, we have:

$$P\left[\bar{m}(F) < \overline{VaR}_\infty(\alpha)\right] = \alpha \quad \Leftrightarrow \quad P\left[F < \bar{m}^{-1}\left(\overline{VaR}_\infty(\alpha)\right)\right] = \alpha,$$

and we deduce that the VaR at population level is:

$$\overline{VaR}_\infty(\alpha) = \bar{m}[Q_\infty(\alpha)] = \sum_{k=1}^{K} \pi_k m_k[Q_\infty(\alpha)] = \sum_{k=1}^{K} \pi_k VaR_{k,\infty}(\alpha), \tag{a.4}$$

by definition of the disaggregated VaR. We have the perfect aggregation formula for the CSA risk measure, that is, the aggregate VaR is obtained by summing the disaggregated VaR with the subpopulation weights.

ii) Aggregation of Granularity Adjustments

We have the following result:

Lemma a.4: *The aggregate granularity adjustment is given by:*

$$\begin{aligned}\overline{GA}(\alpha) &= -\frac{1}{2}\left\{\left(\frac{d\log g_\infty}{dF}[Q_\infty(\alpha)]\frac{1}{\sum_{k=1}^K \pi_k \frac{dm_k}{dF}[Q_\infty(\alpha)]} - \frac{\sum_{k=1}^K \pi_k \frac{d^2 m_k}{dF^2}[Q_\infty(\alpha)]}{\left(\sum_{k=1}^K \pi_k \frac{dm_k}{dF}[Q_\infty(\alpha)]\right)^2}\right)\right.\\ &\quad\cdot\left(\sum_{k=1}^K \pi_k \sigma_k^2[Q_\infty(\alpha)] + \sum_{k=1}^K \pi_k\left(m_k[Q_\infty(\alpha)] - \bar{m}[Q_\infty(\alpha)]\right)^2\right)\\ &\quad\left.+\frac{\sum_{k=1}^K \pi_k \frac{d\sigma_k^2}{dF}[Q_\infty(\alpha)] + \frac{d}{dF}\left\{\sum_{k=1}^K \pi_k\left(m_k(F)-\bar{m}(F)\right)^2\right\}_{F=Q_\infty(\alpha)}}{\sum_{k=1}^K \pi_k \frac{dm_k}{dF}[Q_\infty(\alpha)]}\right\}.\end{aligned}$$

Proof: Let us compute the different terms in the GA formula in Proposition 1 by using the population conditional mean and variance components given in (a.2) and (a.3).

i) The conditional expectation of $\bar{\sigma}^2(F)$ given $\bar{m}(F)$ is:

$$E[\bar{\sigma}^2(F)|\bar{m}(F)=z] = E[\bar{\sigma}^2(F)|F=\bar{m}^{-1}(z)] = \bar{\sigma}^2[\bar{m}^{-1}(z)].$$

Then, from (a.4) we get:

$$\begin{aligned}& E[\bar{\sigma}^2(F)|\bar{m}(F)=\overline{VaR}_\infty(\alpha)]\\ =\;& \bar{\sigma}^2[Q_\infty(\alpha)] = \sum_{k=1}^K \pi_k\sigma_k^2[Q_\infty(\alpha)] + \sum_{k=1}^K \pi_k\left(m_k[Q_\infty(\alpha)] - \bar{m}[Q_\infty(\alpha)]\right)^2,\end{aligned} \tag{a.5}$$

and similarly:

$$E[\sigma_k^2(F)|m_k(F)=VaR_{k,\infty}(\alpha)] = \sigma_k^2[Q_\infty(\alpha)]. \tag{a.6}$$

ii) The derivative of the conditional expectation of $\bar{\sigma}^2(F)$ given $\bar{m}(F)=z$ w.r.t. z is given by:

$$\frac{\partial}{\partial z}E[\bar{\sigma}^2(F)|\bar{m}(F)=z] = \frac{\partial}{\partial z}\bar{\sigma}^2[\bar{m}^{-1}(z)] = \frac{d\bar{\sigma}^2}{dF}[\bar{m}^{-1}(z)]\frac{1}{\frac{d\bar{m}}{dF}[\bar{m}^{-1}(z)]}.$$

Thus:

$$\left.\frac{\partial}{\partial z}E[\bar{\sigma}^2(F)|\bar{m}(F)=z]\right|_{z=\overline{VaR}_\infty(\alpha)} = \frac{\frac{d\bar{\sigma}^2}{dF}[Q_\infty(\alpha)]}{\frac{d\bar{m}}{dF}[Q_\infty(\alpha)]}. \tag{a.7}$$

Similarly:

$$\left.\frac{\partial}{\partial z}E[\sigma_k^2(F)|m_k(F)=z]\right|_{z=VaR_{k,\infty}(\alpha)} = \frac{\frac{d\sigma_k^2}{dF}[Q_\infty(\alpha)]}{\frac{dm_k}{dF}[Q_\infty(\alpha)]}. \tag{a.8}$$

iii) Let us now derive the distribution $\bar{m}(F)$. We have:

$$P[\bar{m}(F) \leq z] = P[F \leq \bar{m}^{-1}(z)] = G_{\infty}[\bar{m}^{-1}(z)].$$

Thus, the pdf of $\bar{m}(F)$ is given by:

$$\bar{g}_{\infty}(z) = g_{\infty}[\bar{m}^{-1}(z)] \frac{1}{\frac{d\bar{m}}{dF}[\bar{m}^{-1}(z)]}.$$

The derivative of the log-density is:

$$\begin{aligned} \frac{d \log \bar{g}_{\infty}(z)}{dz} &= \frac{d}{dz}\left[\log g_{\infty}[\bar{m}^{-1}(z)] - \log \frac{d\bar{m}}{dF}[\bar{m}^{-1}(z)]\right] \\ &= \left(\frac{d \log \bar{g}_{\infty}(F)}{dF} \frac{1}{\frac{d\bar{m}}{dF}(F)} - \frac{\frac{d^2\bar{m}}{dF^2}(F)}{\left[\frac{d\bar{m}}{dF}(F)\right]^2} \right)_{F=\bar{m}^{-1}(z)}. \end{aligned}$$

Thus, we get:

$$\frac{d \log \bar{g}_{\infty}}{dz}[\overline{VaR}_{\infty}(\alpha)] = \frac{d \log g_{\infty}}{dF}[Q_{\infty}(\alpha)] \frac{1}{\frac{d\bar{m}}{dF}[Q_{\infty}(\alpha)]} - \frac{\frac{d^2\bar{m}}{dF^2}[Q_{\infty}(\alpha)]}{\left(\frac{d\bar{m}}{dF}[Q_{\infty}(\alpha)]\right)^2}. \tag{a.9}$$

Similarly:

$$\frac{d \log g_{k,\infty}}{dz}[VaR_{k,\infty}(\alpha)] = \frac{d \log g_{\infty}}{dF}[Q_{\infty}(\alpha)] \frac{1}{\frac{dm_k}{dF}[Q_{\infty}(\alpha)]} - \frac{\frac{d^2 m_k}{dF^2}[Q_{\infty}(\alpha)]}{\left(\frac{dm_k}{dF}[Q_{\infty}(\alpha)]\right)^2}. \tag{a.10}$$

From (a.5), (a.7) and (a.9), and Proposition 1, the GA in Lemma a.4 follows.

QED

In general, the aggregate GA cannot be written as a function of the subpopulations weights, disaggregate CSA VaR and disaggregate GA, given by [use equations (a.6), (a.8) and (a.10), and Proposition 1]:

$$GA_k(\alpha) = -\frac{1}{2}\left\{ \left(\frac{d \log g_{\infty}}{dF}[Q_{\infty}(\alpha)] - \frac{\frac{d^2 m_k}{dF^2}[Q_{\infty}(\alpha)]}{\frac{dm_k}{dF}[Q_{\infty}(\alpha)]} \right) \frac{\sigma_k^2[Q_{\infty}(\alpha)]}{\frac{dm_k}{dF}[Q_{\infty}(\alpha)]} + \frac{\frac{d\sigma_k^2}{dF}[Q_{\infty}(\alpha)]}{\frac{dm_k}{dF}[Q_{\infty}(\alpha)]} \right\}. \tag{a.11}$$

This shows that the difficulty in aggregating the risk measures is due mainly to the idiosyncratic risk.

Perfect aggregation applies for the GA if the subpopulations are homogeneous w.r.t. the systematic risk component, that is, if functions $m_k(.)$ are independent of k.

Corollary 1: *If $m_k(F)$ is independent of $k = 1, \cdots, K$, we have:*

$$\overline{GA}(\alpha) = \sum_{k=1}^{K} \pi_k GA_k(\alpha).$$

Proof: This is a direct consequence of the GA formula given in Lemma a.4.

When functions $m_k(.)$ differ across the subpopulations, Lemma a.4 implies that the aggregate GA is equal to a weighted average of the disaggregate GA, with modified weights that differ from the population weights, plus some correction terms that account for heterogeneity. As an example, let us suppose that the systematic risk components are linear with different factor loadings across the subpopulations:

$$m_k(F) = \beta_k F, \quad k = 1, \cdots, K,$$

with the normalization $\sum_{k=1}^{K} \pi_k \beta_k = 1$. Then, we have $\bar{m}(F) = F$ and from Lemma a.4 we get:

$$\overline{GA}(\alpha) = \sum_{k=1}^{K} \beta_k \pi_k GA_k(\alpha) - \frac{1}{2}\gamma^2 \left(\frac{d \log g_\infty}{dF}[Q_\infty(\alpha)] Q_\infty(\alpha)^2 + 2Q_\infty(\alpha) \right), \tag{a.12}$$

where $\gamma^2 = \sum_{k=1}^{K} \pi_k (\beta_k - 1)^2$ and:

$$GA_k(\alpha) = -\frac{1}{2\beta_k} \left(\frac{d \log g_\infty}{dF}[Q_\infty(\alpha)] \sigma_k^2[Q_\infty(\alpha)] + \frac{d\sigma_k^2}{dF}[Q_\infty(\alpha)] \right).$$

The first term in (a.12) is a weighted average of disaggregated GA with weights proportional to the products of betas by population frequencies. The second term is an adjustment for beta heterogeneity. It involves the variance of betas γ^2 as heterogeneity measure, multiplied by quantity $-\frac{1}{2}\left(\frac{d \log g_\infty}{dF}[Q_\infty(\alpha)] Q_\infty(\alpha)^2 + 2Q_\infty(\alpha) \right)$, which corresponds to the GA computed with $m(F) = F$ and $\sigma^2(F) = F^2$. When the systematic factor F is Gaussian and the conditional variance functions $\sigma_k^2(F) = \sigma^2$ are constant, independent of k, equation (a.12) corresponds to the GA for the linear static RFM with discrete beta heterogeneity (see Example 4.3).

APPENDIX 3: GA when $m(F)$ is monotone increasing

From Proposition 1 we have:

$$
\begin{aligned}
GA(\alpha) &= -\frac{1}{2}\left(\frac{1}{g_\infty(z)}\frac{d}{dz}\left\{g_\infty(z)E\left[\sigma^2(F)|m(F)=z\right]\right\}\right)_{z=VaR_\infty(\alpha)} \\
&= -\frac{1}{2}\left(\frac{1}{g_\infty(z)}\frac{d}{dz}\left\{g_\infty(z)\sigma^2[m^{-1}(z)]\right\}\right)_{z=VaR_\infty(\alpha)} \\
&= -\frac{1}{2}\left(\frac{\frac{dm}{dF}[m^{-1}(z)]}{\tilde{g}_\infty[m^{-1}(z)]}\frac{d}{dz}\left\{\frac{\tilde{g}_\infty[m^{-1}(z)]\sigma^2[m^{-1}(z)]}{\frac{dm}{dF}[m^{-1}(z)]}\right\}\right)_{z=VaR_\infty(\alpha)} \\
&= -\frac{1}{2}\left(\frac{1}{\tilde{g}_\infty[m^{-1}(z)]}\frac{d}{dm^{-1}(z)}\left\{\frac{\tilde{g}_\infty[m^{-1}(z)]\sigma^2[m^{-1}(z)]}{\frac{dm}{dF}[m^{-1}(z)]}\right\}\right)_{z=VaR_\infty(\alpha)} \\
&= -\frac{1}{2}\left(\frac{1}{\tilde{g}_\infty(f)}\frac{d}{df}\left\{\frac{\tilde{g}_\infty(f)\sigma^2(f)}{\frac{dm}{dF}(f)}\right\}\right)_{f=m^{-1}[VaR_\infty(\alpha)]} .
\end{aligned}
$$

APPENDIX 4: Granularity adjustment in the Merton-Vasicek model

In this Appendix we prove equation (4.9) for the GA in the Merton Vasicek model. From (4.4)-(4.7) we get:

$$
GA(\alpha) = -\frac{1}{2}\left\{\frac{\frac{\Phi^{-1}(PD)}{\sqrt{1-\rho}} - \frac{1-2\rho}{\sqrt{\rho(1-\rho)}}\Phi^{-1}(\alpha)}{\phi\left(\frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)}VaR_\infty(\alpha)[1-VaR_\infty(\alpha)] + 1 - 2VaR_\infty(\alpha)\right\}.
$$

Now, we have:

$$
\begin{aligned}
\frac{\Phi^{-1}(PD)}{\sqrt{1-\rho}} - \frac{1-2\rho}{\sqrt{\rho(1-\rho)}}\Phi^{-1}(\alpha) &= \frac{\Phi^{-1}(PD)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}} - \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(\alpha) \\
&= \Phi^{-1}\left[VaR_\infty(\alpha)\right] - \sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(\alpha).
\end{aligned}
$$

Then, equation (4.9) follows.

APPENDIX 5: Stochastic probability of default and expected loss given default

In this Appendix we give a detailed derivation of the granularity adjustment in the model with stochastic probability of default and expected loss given default presented in Section 6.

i) ELGD and VLGD

The unconditional expected loss given default is:

$$\begin{aligned} ELGD &= E[LGD_{i,t}|Z_{i,t}=1] = E[E[LGD_{i,t}|F_t, Z_{i,t}=1]|Z_{i,t}=1] = E[F_{2,t}|Z_{i,t}=1] \\ &= \frac{E[F_{2,t}\mathbb{1}_{Z_{i,t}=1}]}{E[\mathbb{1}_{Z_{i,t}=1}]} = \frac{E[F_{2,t}E[Z_{i,t}|F_t]]}{E[Z_{i,t}]} = \frac{E[F_{2,t}F_{1,t}]}{E[F_{1,t}]} = E^*[F_{2,t}], \end{aligned}$$

where $E^*[\cdot]$ denotes expectation w.r.t. the probability distribution defined by the change of measure $F_{1,t}/E[F_{1,t}]$. The unconditional variance of the loss given default is:

$$\begin{aligned} VLGD &= V[LGD_{i,t}|Z_{i,t}=1] = E[LGD_{i,t}^2|Z_{i,t}=1] - ELGD^2 \\ &= E[E[LGD_{i,t}^2|F_t, Z_{i,t}=1]|Z_{i,t}=1] - ELGD^2 \\ &= E[V[LGD_{i,t}|F_t]|Z_{i,t}=1] + E[E[LGD_{i,t}|F_t]^2|Z_{i,t}=1] - ELGD^2 \\ &= \gamma E[F_{2,t}(1-F_{2,t})|Z_{i,t}=1] + E[F_{2,t}^2|Z_{i,t}=1] - ELGD^2 \\ &= (1-\gamma)E[F_{2,t}^2|Z_{i,t}=1] + ELGD(\gamma - ELGD) \\ &= (1-\gamma)\frac{E[F_{2,t}^2F_{1,t}]}{E[F_{1,t}]} + ELGD(\gamma - ELGD) \\ &= \gamma ELGD(1-ELGD) + (1-\gamma)\left(\frac{E[F_{2,t}^2F_{1,t}]}{E[F_{1,t}]} - ELGD^2\right). \end{aligned}$$

Thus we get:

$$VLGD = \gamma ELGD(1-ELGD) + (1-\gamma)V^*[F_{2,t}].$$

ii) Cross-sectional factor approximation

Let us consider date t and assume that we observe the individual losses $y_{i,t}$, $i=1,\cdots,n$, of the zero-coupon corporate bonds maturing at date t. Equivalently, we observe the default indicator $Z_{i,t} = \mathbb{1}_{y_{i,t}>0}$ for all individual companies, and the Loss Given Default $LGD_{i,t} = y_{i,t}$, if $Z_{i,t}=1$. Thus, the model is equivalent to a tobit model and the cross-sectional likelihood conditional on the factor value is:

$$\begin{aligned} \prod_{i=1}^{n} h(y_{i,t}|f_t) &= \left[\prod_{i=1}^{n} f_{1,t}^{Z_{i,t}}(1-f_{1,t})^{1-Z_{i,t}}\right] \prod_{i:Z_{i,t}=1} \frac{\Gamma(a_t+b_t)}{\Gamma(a_t)\Gamma(b_t)} LGD_{i,t}^{a_t-1}(1-LGD_{i,t})^{b_t-1} \\ &= f_{1,t}^{N_t}(1-f_{1,t})^{n-N_t}\left(\frac{\Gamma[(1-\gamma)/\gamma]}{\Gamma(a_t)\Gamma(b_t)}\right)^{N_t}\left[\prod_{i:y_{i,t}>0} y_{i,t}\right]^{a_t-1}\left[\prod_{i:y_{i,t}>0}(1-y_{i,t})\right]^{b_t-1}, \end{aligned}$$

where $N_t = \sum_{i=1}^{n} \mathbb{1}_{y_{i,t}>0}$ is the number of defaults at date t, and $a_t = \left(\frac{1-\gamma}{\gamma}\right) f_{2,t}$ and $b_t = \left(\frac{1-\gamma}{\gamma}\right)(1-f_{2,t})$. We get the cross-sectional log-likelihood:

$$
\begin{aligned}
\sum_{i=1}^{n} \log h(y_{i,t}|f_t) &= N_t \log f_{1,t} + (n-N_t)\log(1-f_{1,t}) + N_t \log \Gamma\left[\frac{1-\gamma}{\gamma}\right] \\
&\quad - N_t \log \Gamma\left[\left(\frac{1-\gamma}{\gamma}\right) f_{2,t}\right] - N_t \log\Gamma\left[\left(\frac{1-\gamma}{\gamma}\right)(1-f_{2,t})\right] \\
&\quad + \left[\left(\frac{1-\gamma}{\gamma}\right) f_{2,t} - 1\right] \sum_{i:y_{i,t}>0} \log y_{i,t} + \left[\left(\frac{1-\gamma}{\gamma}\right)(1-f_{2,t}) - 1\right] \sum_{i:y_{i,t}>0} \log(1-y_{i,t}) \\
&= \mathcal{L}_{1,n,t}(f_{1,t}) + \mathcal{L}_{2,n,t}(f_{2,t}), \quad \text{say.}
\end{aligned} \tag{a.13}
$$

which is decomposed as the sum of a function of $f_{1,t}$ and a function of $f_{2,t}$. Therefore, the cross-sectional approximations of the two factors can be computed separately, and we get (6.1) and (6.2).

iii) Approximate filtering distribution of f_t given $I_{n,t}$

From a multiple factor version of Corollary 5.3 in Gagliardini, Gouriéroux (2010a), and the log-likelihood decomposition in (a.13), it follows that the approximate filtering distribution is such that $F_{1,t}$ and $F_{2,t}$ are independent conditional on information $I_{n,t}$ at order $1/n$, with Gaussian distributions $N\left(\hat{f}_{l,n,t} + \frac{1}{n}\mu_{l,n,t}, \frac{1}{n} J_{l,n,t}^{-1}\right)$, for $l = 1, 2$, where:

$$
\begin{aligned}
\mu_{l,n,t} &= J_{l,n,t}^{-1} \frac{\partial \log g}{\partial f_{l,t}}(\hat{f}_{n,t}|\hat{f}_{n,t-1}) + \frac{1}{2} J_{l,n,t}^{-2} K_{l,n,t}, \\
J_{l,n,t} &= -\frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2 \log h}{\partial f_{l,t}^2}(y_{i,t}|\hat{f}_{n,t}) = -\frac{1}{n}\frac{\partial^2 \mathcal{L}_{l,n,t}}{\partial f_{l,t}^2}(\hat{f}_{l,n,t}), \\
K_{l,n,t} &= \frac{1}{n}\sum_{i=1}^{n} \frac{\partial^3 \log h}{\partial f_{l,t}^3}(y_{i,t}|\hat{f}_{n,t}) = \frac{1}{n}\frac{\partial^3 \mathcal{L}_{l,n,t}}{\partial f_{l,t}^3}(\hat{f}_{l,n,t}),
\end{aligned}
$$

and where g denotes the transition pdf of factor F_t. From (a.13), equations (6.4) and (6.5) follow, as well as $K_{1,n,t} = 2\frac{1-2\hat{f}_{1,n,t}}{\hat{f}_{1,n,t}^2(1-\hat{f}_{1,n,t})^2}$. Moreover:

$$
\begin{aligned}
\frac{\partial \log g}{\partial f_{l,t}}(f_t|f_{t-1}) &= -\frac{1}{a[A^{-1}(f_{l,t})]}\left\{e_l'\left[\Omega^{-1}(f_t^* - c - \Phi f_{t-1}^*)\right] + \frac{d\log a}{dy}[A^{-1}(f_{l,t})]\right\} \\
&= -\frac{1}{f_{l,t}(1-f_{l,t})}\left\{e_l'\left[\Omega^{-1}(f_t^* - c - \Phi f_{t-1}^*)\right] + 1 - 2f_{l,t}\right\},
\end{aligned}
$$

for $l = 1, 2$, where $A(y) = [1+\exp(-y)]^{-1}$ and $a(y) = dA(y)/dy$. Then equations (6.3) follow.

iv) A useful lemma

The derivation of the CSA VaR and the GA below uses the next lemma.

Lemma a.5: *Let X and Y be two random variables on $[0,1]$. Denote by $f(x,y)$, $f_2(y)$, $F_2(y)$ and $F_{1|2}(x|y)$ the joint pdf of (X,Y), the pdf of Y, the cdf of Y and the conditional cdf of X given $Y=y$, respectively. Let $Z = XY$. Then:*

(i) $P[Z \leq z] = \int_z^1 F_{1|2}(z/y|y) f_2(y) dy + F_2(z)$, *for any* $z \in [0,1]$;

(ii) *The pdf of Z is* $g(z) = \int_z^1 \frac{1}{y} f(z/y, y) dy$, $z \in [0,1]$;

(iii) $E[Y|Z=z] = \dfrac{\int_z^1 f(z/y,y)dy}{\int_z^1 \frac{1}{y} f(z/y,y)dy}$, *for any* $z \in [0,1]$.

Proof: (i) We have:

$$P[Z \leq z] \;=\; EP[Z \leq z|Y] = EP[X \leq z/Y|Y].$$

Now, $P[X \leq z/Y|Y] = \mathbb{1}_{z \leq Y} F_{1|2}(z/Y|Y) + \mathbb{1}_{z > Y}$. Thus, we get:

$$P[Z \leq z] = \int_z^1 F_{1|2}(z/y|y) f_2(y) dy + F_2(z).$$

(ii) By differentiating the cdf found in (i) we get:

$$\begin{aligned} g(z) &= \frac{d}{dz} \left(\int_z^1 F_{1|2}(z/y|y) f_2(y) dy + F_2(z) \right) \\ &= \int_z^1 \frac{1}{y} f(z/y|y) f_2(y) dy - \left[F_{1|2}(z/y|y) f_2(y) \right]_{y=z} + f_2(z) = \int_z^1 \frac{1}{y} f(z/y,y) dy. \end{aligned}$$

(iii) Let us consider the change of variables from (x,y) to (z,y), where $z = xy$. The Jacobian is $\left| \det \left[\frac{\partial(z,y)}{\partial(x,y)} \right] \right| = y$. Thus, the joint density of Z and Y is $g(z,y) = \frac{1}{y} f(z/y, y)$, for $0 \leq z \leq y \leq 1$. We get:

$$E[Y|Z=z] = \frac{\int_z^1 y g(z,y) dy}{\int_z^1 g(z,y) dy} = \frac{\int_z^1 f(z/y,y) dy}{\int_z^1 \frac{1}{y} f(z/y,y) dy}.$$

QED

v) CSA risk measure

Let us first compute function $a(w, 0; f)$. We have:

$$m(F_{t+1}) = E[LGD_{i,t+1}|F_{t+1}] E[Z_{i,t+1}|F_{t+1}] = F_{1,t+1} F_{2,t+1}. \tag{a.14}$$

Then, from (5.6):

$$a(w,0;f_t) = P[F_{1,t+1}F_{2,t+1} \leq w | F_t = f_t].$$

Let us apply Lemma a.5 (i) with $X = F_{2,t+1}$ and $Y = F_{1,t+1}$, conditionally on $F_t = f_t$. The distribution of $F^*_{2,t+1}$ conditional on $F^*_{1,t+1}$ and $F_t = f_t$ is:

$$N\left(c_{2,t} + \frac{\rho\sigma_2}{\sigma_1}(F^*_{1,t+1} - c_{1,t}), \sigma_2^2(1-\rho^2)\right),$$

where $c_{1,t} = c_1 + \Phi_{11}A^{-1}(f_{1,t}) + \Phi_{12}A^{-1}(f_{2,t})$ and $c_{2,t} = c_2 + \Phi_{21}A^{-1}(f_{1,t}) + \Phi_{22}A^{-1}(f_{2,t})$. Thus, the conditional cdf of $F_{2,t+1}$ given $F_{1,t+1} = y$ and $F_t = f_t$ is:

$$P[F_{2,t+1} \leq x | F_{1,t+1} = y, F_t = f_t] = \Phi\left[\frac{A^{-1}(x) - c_{2,t} - \frac{\rho\sigma_2}{\sigma_1}(A^{-1}(y) - c_{1,t})}{\sigma_2\sqrt{1-\rho^2}}\right],$$

for $x, y \in (0,1)$. The cdf and the pdf of $F_{1,t+1}$ conditional on $F_t = f_t$ are:

$$F(y|f_t) = \Phi\left(\frac{A^{-1}(y) - c_{1,t}}{\sigma_1}\right), \quad f(y|f_t) = \frac{1}{\sigma_1}\varphi\left(\frac{A^{-1}(y) - c_{1,t}}{\sigma_1}\right)\frac{1}{a[A^{-1}(y)]},$$

respectively, where $a(y) = dA(y)/dy$. Thus, from Lemma a.5 we get:

$$\begin{aligned} a(w,0;f_t) &= \Phi\left(\frac{A^{-1}(w) - c_{1,t}}{\sigma_1}\right) + \int_w^1 \Phi\left[\frac{A^{-1}(w/y) - c_{2,t} - \frac{\rho\sigma_2}{\sigma_1}(A^{-1}(y) - c_{1,t})}{\sigma_2\sqrt{1-\rho^2}}\right] \\ &\qquad \cdot\frac{1}{\sigma_1}\varphi\left(\frac{A^{-1}(y) - c_{1,t}}{\sigma_1}\right)\frac{1}{a[A^{-1}(y)]}dy. \end{aligned}$$

By using that $a[A^{-1}(y)] = y(1-y)$ and $A^{-1}(y) = \log[y/(1-y)]$, Proposition 5 i) follows.

vi) Granularity adjustment

Let us first compute the conditional density $g_\infty(.;f_t)$ of $m(F_{t+1})$ given $F_t = f_t$. We use equation (a.14) and apply Lemma a.5 (ii) with $X = F_{1,t+1}$ and $Y = F_{2,t+1}$ conditionally on $F_t = f_t$. The joint density of (X,Y) conditionally on $F_t = f_t$ is:

$$f(x,y) = \frac{1}{\sigma_1\sigma_2}\varphi\left(\frac{A^{-1}(x) - c_{1,t}}{\sigma_1}, \frac{A^{-1}(y) - c_{2,t}}{\sigma_2}; \rho\right)\frac{1}{a[A^{-1}(x)]a[A^{-1}(y)]},$$

where $\varphi(.,.;\rho)$ denotes the pdf of a bivariate standard Gaussian distribution with correlation parameter ρ. We get:

$$g_\infty(w;f_t) = \int_w^1 \frac{1}{y}\frac{1}{\sigma_1\sigma_2}\varphi\left(\frac{A^{-1}(w/y) - c_{1,t}}{\sigma_1}, \frac{A^{-1}(y) - c_{2,t}}{\sigma_2}; \rho\right)\frac{1}{a[A^{-1}(w/y)]a[A^{-1}(y)]}dy.$$

Let us now compute $E[\sigma^2(F_{t+1})|m(F_{t+1})=w,F_t=f_t]$. We have:

$$\begin{aligned}\sigma^2(F_{t+1}) &= E[LGD_{i,t+1}^2|F_{t+1}]E[Z_{i,t+1}|F_{t+1}]-E[LGD_{i,t+1}|F_{t+1}]^2E[Z_{i,t+1}|F_{t+1}]^2\\ &= \gamma F_{2,t+1}(1-F_{2,t+1})F_{1,t+1}+F_{1,t+1}(1-F_{1,t+1})F_{2,t+1}^2,\end{aligned}$$

By replacing $F_{1,t+1}F_{2,t+1}$ with $m(F_{t+1})$ in the above equation, function $\sigma^2(F_{t+1})$ can be rewritten as:

$$\begin{aligned}\sigma^2(F_{t+1}) &= \gamma(F_{1,t+1}F_{2,t+1})(1-F_{2,t+1})+(F_{1,t+1}F_{2,t+1})(F_{2,t+1}-F_{1,t+1}F_{2,t+1})\\ &= m(F_{t+1})[\gamma-m(F_{t+1})]+(1-\gamma)m(F_{t+1})F_{2,t+1}.\end{aligned}$$

Thus:

$$\begin{aligned}E\left[\sigma^2(F_{t+1})|m(F_{t+1})=w,F_t=f_t\right] &= w(\gamma-w)\\ &\quad +(1-\gamma)wE[F_{2,t+1}|F_{1,t+1}F_{2,t+1}=w,F_t=f_t]. \qquad \text{(a.15)}\end{aligned}$$

From Lemma a.5 (iii) with $X=F_{1,t+1}$ and $Y=F_{2,t+1}$, conditionally on $F_t=f_t$, we get:

$$E[F_{2,t+1}|m(F_{t+1})=w,F_t=f_t]=\frac{b(w,1;f_t)}{g_\infty(w;f_t)},$$

where:

$$b(w,1;f_t)=\int_w^1\frac{1}{\sigma_1\sigma_2}\varphi\left(\frac{A^{-1}\left(w/y\right)-c_{1,t}}{\sigma_1},\frac{A^{-1}(y)-c_{2,t}}{\sigma_2};\rho\right)\frac{1}{a[A^{-1}\left(w/y\right)]a[A^{-1}(y)]}dy.$$

Then, from equation (a.15) we get:

$$E[\sigma^2(F_{t+1})|m(F_{t+1})=w,F_t=f_t]=w(\gamma-w)+(1-\gamma)w\frac{b(w,1;f_t)}{g_\infty(w;f_t)}.$$

By using $a[A^{-1}\left(w/y\right)]a[A^{-1}(y)]=w(y-w)(1-y)/y$, Proposition 5 ii) follows.