HULL-WHITE OR
JARROW-TURNBULL ?
THE SAME CREDIT ENGINE

by Gilles Desvilles[1]

I) INTRODUCTION

Several researchers paved the way of the success of the credit markets with their models: Duffie, Merton, KMV … . But so far the two most acclaimed by the practitioners are undoubtedly the Hull-White model and the Jarrow-Turnbull model, probably because of their reasonable degree of complexity and their width of scope.

Both need to derive probabilities of default before being able to price assets and derivatives with credit exposure. Both use a government and a rated yield curve, along with assumptions about recovery in the event of default. Despite these similarities the markets oppose them for two motives: 1) Hull-White handles benchmark bonds bearing coupons whereas Jarrow-Turnbull deals only with zero-coupons; 2) the former does not rely on an interest rate diffusion contrary to the latter. This certainly explains why reconciling the two models has never been published. Yet the present paper unveils the theoretical foundations they both share.

Part II summarizes the credit engines of the two models, recalls the original pricing formulae and emphasizes the two main features believed to distinguish the engines, the Hull-White ability to handle coupon-bearing bonds and the Jarrow-Turnbull requirement to implement a two-factor lattice.

Part III shows that: 1) by revealing its lengthy proof, the Hull-White pricing formula relies extensively on the viable market assumption, and hides traps to the practitioners; 2) the Jarrow-Turnbull credit engine can work without any interest rates diffusion in deriving default probabilities; 3) the Jarrow-Turnbull engine can be adapted without distortion to accept coupon-bonds as inputs; 4) in a basic zero-coupon two-period framework both engines yield the same default probabilities; this result is extended to any number of periods and to coupon-bearing bonds. Hence the two models are equivalent.

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Part IV compares numerically the default probabilities implied by credit rated bonds bearing different coupons. A non desirable feature is that the probabilities for a given rating depends on the coupon level, compelling the need for a unique set of probabilities extracted from a panel of the most liquid bonds. Excepted in a special case, the paper rules out the method of pricing any corporate bond with zero coupon rates bootstrapped from corporate benchmarks. One must instead bootstrap the benchmarks' default probabilities and plug them into any one of the two equivalent credit models.

II) THE HULL-WHITE AND JARROW-TURNBULL CONTEST

The models presented here are voluntarily restricted to the derivation of the default probabilities implied by a set of bond prices. The purpose is not to show how to value a credit risky bond or a credit default swap but to circumvent the innovation of these models when compared with traditional prevailing yield curve and option models: their credit engines. That is, how they do handle the risk of default on debt servicing, as much independently as possible of the other risks like those stemming from interest rate or volatility shifts.

Both engines are recalled in their original setting before paralleling them. Their common goal is to extract default probabilities from selected bonds of same rating.

II.1 The Hull-White Engine[2]

A panel of N most liquid bonds with the same credit risk is designed, and they are named corporate bonds for simplicity. Bond j pays fixed coupons and matures at date t_j. Defaults on any bond j can occur only at discrete times corresponding to the maturities t_i of the shorter bonds or to t_j. Interest rates, either corporate or government, are random. But the recovery rate in case of a default is not and is set to a constant R.

The idea is to compare the price of the corporate bonds with the price of their government equivalents (same maturity and coupons). Their difference contains the default premium. Note that the original model does not use directly a panel of government benchmarks but a default-free term structure in order to price the government equivalents. Hence the maturities t_is are here evenly spaced, typically by one year.

Let B_j be the today price of the corporate bond j and G_j be the today price of its equivalent government bond. Let $F_j(t)$ be the today forward price of the equivalent bond to be delivered at time t and $\upsilon(t)$ the present value of 1 dollar received at t. $C_j(t)$ is what the corporate bond j holders claim when their bond defaults at time t and $RC_j(t)$ is only what they recover on their claim.

Two cases are studied: Case 1 where $C_j(t_i)$ is worth the bond face value plus accruals, which is a constant, and Case 2 where $C_j(t_i)$ is worth $G_j(t_i)$.

Recall that the corporate bonds default only on t_i's. On such a date, the corporate bond is recovered at a value $RC_j(t_i)$ and its equivalent government bond is worth $G_j(t_i)$. Such an event has a probability p_i of occurrence. The relative loss is then $G_j(t_i) - RC_j(t_i)$. Its present value in the risk-free world is noted α_{ij} and uses the default-free discount factor:

$$\alpha_{ij} = \upsilon(t_i)\ [G_j(t_i) - RC_j(t_i)]$$

Then Hull and White assert that in the risk-neutral world:

$$\begin{cases} G_j(t_i) = F_j(t_i) \\ G_j - B_j = \sum_{i=1}^{i=j} p_i\ \alpha_{ij} \qquad \text{(``Hull White pricing formula'')} \end{cases} \tag{1}$$

The implied probabilities p_i's are then bootstrapped, starting with p_1 and ending with p_N, through the formula

$$p_j = \frac{G_j - B_j - \sum_{i=1}^{i=j-1} p_i \alpha_{ij}}{\alpha_{jj}} \tag{2}$$

The assertion in parenthesis is the credit engine per se. But the two authors did not formally prove it and we shall see in section III.1 that so doing reveals that their hypotheses need be clarified and chiefly their formula be amended.

II.2 The Jarrow-Turnbull Engine[3]

The model is presented as a two-factor lattice, the first factor being the default-free short-term interest rate, the second being a corporate bankruptcy proxy worth either the recovery rate or 1.

Jarrow and Turnbull interpret this proxy as a fictitious exchange rate and model it as such. It turns out that they do not need to make a direct use of this

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[2] ''Valuing Credit Default Swaps I: No Counterparty Default Risk'', John Hull and Alan White, Journal of Derivatives, Fall 2000, and ''Options, futures, & other derivatives'', chapter 26, John C. Hull, 5th international edition, Prentice Hall finance series, 2003.

[3] ''Pricing Options on Derivative Securities Subject to Credit Risk'', Robert Jarrow and Stuart Turnbull, Journal of Finance, 1995, and ''Derivative Securities'', 2nd edition, chapter 6, Robert Jarrow and Stuart Turnbull, South-Western, 2000.

factor when getting their main result and following this observation the paper will leave it implicit.

By a change of probability the world becomes risk-neutral and in this setting the probability π of a rise of the short rate is supposed independent of the probability λ of default of the corporate bond. π is stationary but λ depends on time, and π is unconditional whereas $\lambda(t)$ is conditional upon no default before t.

The short rate is denoted $r_X(t)$ where t stands for the current time and X for the state of nature, for example X = UD means the rate has first gone up from t = 0 to t = 1 then down from t = 1 to t=2.

A set of N corporate bonds is used to get the implied default probabilities $\lambda(t)$ of the class of risk to which they belong. All these bonds are zero-coupons with different maturities evenly spaced by one period. The spot price of one such bond is denoted $\upsilon(0,T)$, its future price at time t being $\upsilon(t,T)$. When the obligor defaults then the bond is partially redeemed δ at maturity T, else the bond pays 1 dollar at T.

For example, the lattice for the three-period bond is as follows:

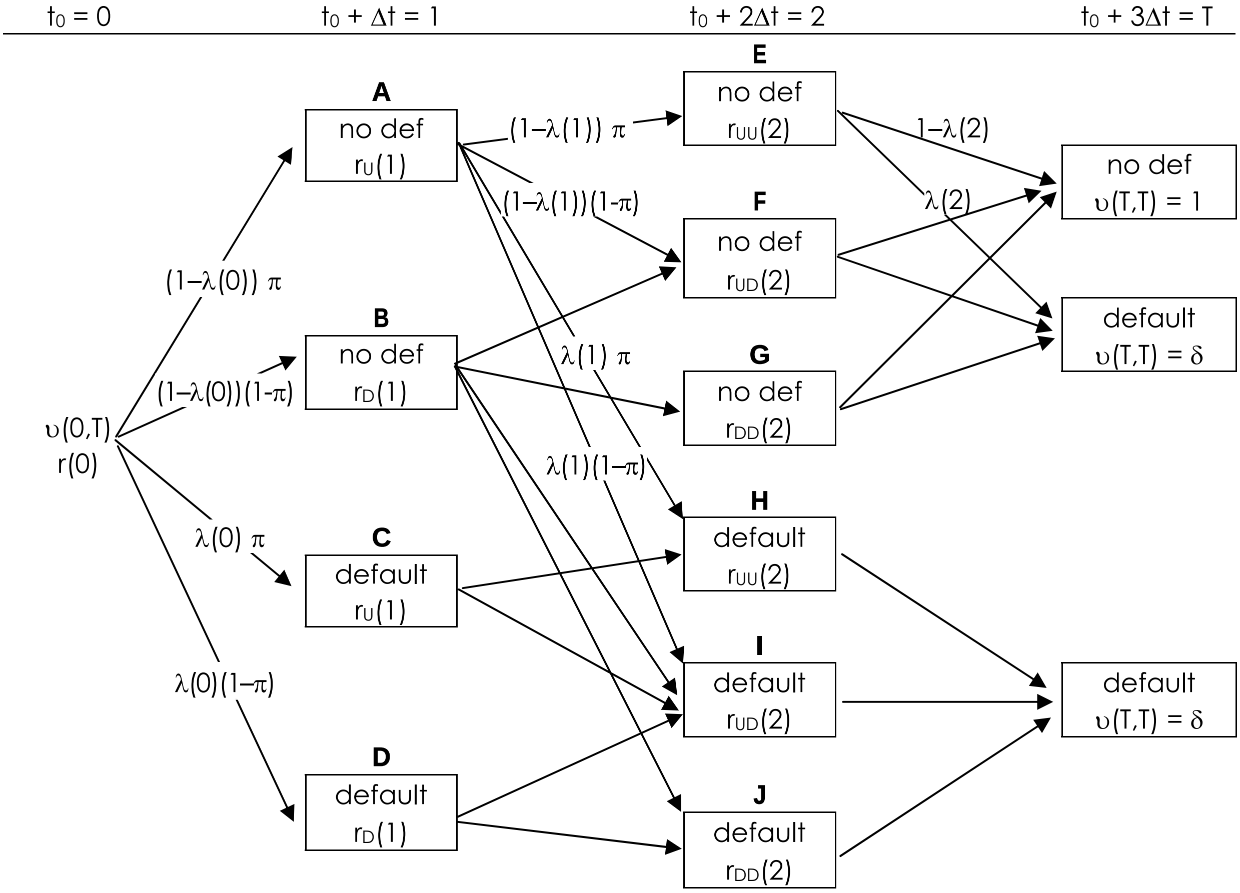

Diagram 1

The lattice is recombining because the short-rate tree is, and because all the recovery proceeds are paid down at maturity whatever the first default date.

The pricing of the corporate bond is done recursively using the martingale equality. For example:

$$\upsilon(0,3) = E[e^{-r(0)}\ \upsilon(1,3)]$$

$$= e^{-r(0)}\ [(1-\lambda(0))\ \pi\ \upsilon_A(1,3) + (1-\lambda(0))\ (1-\pi)\ \upsilon_B(1,3) + \lambda(0)\ \pi\ \upsilon_C(1,3) + \lambda(0)\ (1-\pi)\ \upsilon_D(1,3)]$$

Then $\upsilon_A(1,3)$ is expressed in terms of $\lambda(1)$:

$$e^{-r(0)}\ \upsilon_A(1,3) = E[e^{-r(0)}e^{-r(1)}\ \upsilon(2,3)\ \mid \text{state A at } t = 1]$$

$$\upsilon_A(1,3) = E[e^{-r(1)}\ \upsilon(2,3)\ \mid \text{state A at } t = 1]$$

$$\upsilon_A(1,3) = e^{-r_U(1)}\ [(1-\lambda(1))\ \pi\ \upsilon_E(2,3) + (1-\lambda(1))\ (1-\pi)\ \upsilon_F(2,3) + \lambda(1)\ \pi\ \upsilon_H(2,3) + \lambda(1)\ (1-\pi)\ \upsilon_I(2,3)]$$

The same is done for $\upsilon_B(1,3)$, $\upsilon_C(1,3)$ and $\upsilon_D(1,3)$ so that there now exists a linear form between the corporate bond price $\upsilon(0,3)$ and the first two default probabilities $\lambda(0)$ and $\lambda(1)$.

The node A contribution to the pricing process ends up by expressing $\upsilon_E(2,3)$, $\upsilon_F(2,3)$, $\upsilon_H(2,3)$ and $\upsilon_I(2,3)$ as a function of $\lambda(2)$. For example:

$$e^{-r(0)}e^{-r_U(1)}\ \upsilon_E(2,3) = E[e^{-r(0)}e^{-r_U(1)}e^{-r(2)}\ \upsilon(3,3)\ \mid \text{state E at } t = 2]$$

$$\upsilon_E(2,3) = E[e^{-r(2)}\ \upsilon(3,3)\ \mid \text{state E at } t = 2] = e^{-r_{UU}(2)}\ [(1-\lambda(2)) + \lambda(2)\ \delta]$$

Repeating this work for nodes B, C and D finally delivers a linear form between the current bond price $\upsilon(0,3)$ and the first three default probabilities $\lambda(0)$, $\lambda(1)$ and $\lambda(2)$.

If one knows already $\lambda(0)$ and $\lambda(1)$ this relation allows to extract $\lambda(2)$ from $\upsilon(0,3)$. $\lambda(0)$ itself is extracted by using one such relation featuring $\upsilon(0,1)$, the price of the one-year zero-coupon corporate bond in the one period case. In turn $\lambda(1)$ is extracted by using the just found $\lambda(0)$ and the relation between $\upsilon(0,2)$, $\lambda(0)$, and $\lambda(1)$ in the two period case. Therefore it appears that the default probabilities are bootstrapped, starting from $\lambda(0)$ and ending at $\lambda(N)$.

II.3 The Contest between the Two Engines

Practitioners often oppose the two engines because the Hull-White can handle coupon-bearing corporate bonds and thus be fed with corporate benchmark issues whereas the Jarrow-Turnbull cannot and needs an additional step to extract zero-coupons from the benchmarks. Moreover the former uses only spot bond prices whereas the latter requires a tedious two-factor-lattice construction.

Clearly so far most market participants prefer the Hull-White engine according to its seemingly greater convenience and input relevance.

III) THE HULL-WHITE AND JARROW-TURNBULL RECONCILIATION

This part scrutinizes the alleged advantages and pitfalls of the two credit engines and ends up bringing them together.

III.1 Proof and Adjustments of the Hull-White Formula

In their article Hull and White do not prove formally their pricing formula. They deal with a vague risk-neutral world, do not invoke explicitly any martingale property and do not relax their constant interest rate hypothesis. This section sets up a proof that will unveil sufficient conditions and arrive at some necessary modifications of the formula.

III.1.1 The Zero-Coupon Bonds Case

From now on we assume that the market is viable in Harrison-Pliska sense[4], so there exists at least one risk-neutral probability under which the discounted prices of all no cash flow paying assets are martingales, which means for two zero-coupon bonds, one corporate and the other being its equivalent governmental:

$$E[\upsilon(t_i)\ G_j(t_i)] = G_j \qquad \text{and} \qquad E[\upsilon(t_i)\ B_j(t_i)] = B_j \tag{3}$$

Note that the discount factor $\upsilon(t_i)$ is a known quantity at current time and can be taken out of the expectation operator.

When the value $B_j(t_i)$ of the corporate bond is worth $RC_j(t_i)$ this means that default has occurred at t_i. Assuming independence between default and risk-free rates as did Hull and White we have:

$$E[\upsilon(t_i)\ RC_j(t_i)\ 1\{\text{default at } t_i\}] = E[\upsilon(t_i)\ RC_j(t_i)] \times P(\text{default at } t_i)$$

Using these relations at time t_j (i.e. maturity) for bond j we get:

$$E[\upsilon(t_j)\ G_j(t_j)] = E[\upsilon(t_j)] = G_j$$

and

$$E[\upsilon(t_j)\ B_j(t_j)] = B_j$$

Now consider as an asset the bond likely to default in any t_i whose recovery proceeds are reinvested up to maturity t_j in a \$1 face value zero-coupon government bond worth $\upsilon(t_i,t_j)$, a random price until t_i. Denote it $\tilde{B}$. Then:

$$E[\upsilon(t_j)\ \tilde{B}_j(t_j)] = \sum_{i=1}^{i=j} E[\upsilon(t_j)\ RC_j(t_i)\ \upsilon^{-1}(t_i,t_j)\ 1\{\text{first default at } t_i\}] + E[\upsilon(t_j)\ 1\{\text{first default} > t_j\}]$$

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[4] "Martingales and Stochastic Integrals in the Theory of Continuous Trading", J. M. Harrison and S. R. Pliska,

$$= \sum_{i=1}^{i=j} E[\upsilon(t_j)\ RC_j(t_i)\ \upsilon^{-1}(t_i,t_j)] \times P(\text{first default at } t_i) + E[\upsilon(t_j)] \times P(\text{first default} > t_j) \quad (4)$$

Note that the generic unconditional probability in the above formula requires no default before t_i so that the events {first default at t_i} are mutually independent. Remind these events are also assumed independent of the interest rates.

Applying twice the martingale property for the recovery proceeds reinvested in zero coupon bond υ until maturity, once at time t_i and once at time t_j, we have:

$$\text{Current price of recovery proceeds} = E[\upsilon(t_i)\ RC_j(t_i)] = E[\upsilon(t_j)\ RC_j(t_i)\ \upsilon^{-1}(t_i,t_j)] \quad (5)$$

and denoting p_i the probability P(first default in t_i) the previous equation becomes:

$$\tilde{B}_j = E[\upsilon(t_j)\ \tilde{B}_j(t_j)] = \sum_{i=1}^{i=j} p_i\ E[\upsilon(t_i)\ RC_j(t_i)] + (1-\sum_{i=1}^{i=j} p_i)\ E[\upsilon(t_j)] \quad (6)$$

$\tilde{B}_j = B_j$ as we start reinvesting cash outflows from an asset worth B_j.

Now rewrite G_j as $(\sum_{i=1}^{i=j} p_i + 1-\sum_{i=1}^{i=j} p_i)\ G_j$, and use the martingale property again, $G_j = E[\upsilon(t_1)\ G_j(t_1)] = E[\upsilon(t_2)\ G_j(t_2)] = \ldots = E[\upsilon(t_j)]$, then:

$$G_j = \sum_{i=1}^{i=j} p_i\ E[\upsilon(t_i)\ G_j(t_i)] + (1-\sum_{i=1}^{i=j} p_i)\ E[\upsilon(t_j)] \quad (7)$$

Collecting all terms we find:

$$G_j - B_j = \sum_{i=1}^{i=j} p_i\ E[\upsilon(t_i)\ (G_j(t_i) - RC_j(t_i))] \quad (8)$$

As $\upsilon(t_i)$ is non random at the current time, it can be taken out of the expectation which brings out the risk-neutral pricing formula:

$$G_j - B_j = \sum_{i=1}^{i=j} p_i\ \upsilon(t_i)\ (E[G_j(t_i) - RC_j(t_i)]) \quad (8')$$

The main hypothesis underlying this result is that the market be viable. It allows to use any discounted asset as a martingale, and that is what has been done here with capitalized bond recovery proceeds. The change of numéraire theory[5] where the new numéraire would be the government zero-coupon bond maturing at t_j also provides the same martingale property. However our result cannot rely on this theory because it requires that the assets follow Ito

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Stochastic Processes and their Applications, 1981.

[5] ''Stochastic Calculus for Finance II'', page 393, Steven Schreve, Springer Finance, 2004.

processes with the same market price of risk. Such an assumption would not make sense here and it does not feature in the Hull-White framework.

This result is close to but not strictly equal to the ''Hull-White pricing formula'':

$$G_i - B_i = \sum_{j=1}^{j=i} p_j \, \upsilon(t_j) \, (F_i(t_j) - RC_i(t_j)) \qquad (1')$$

The differences are $E[G_j(t_i)]$ instead of $F_j(t_i)$ and $E[RC_j(t_i)]$ instead of $RC_j(t_i)$. Could we have $E[G_j(t_i)] = F_j(t_i)$? $E[RC_j(t_i)] = RC_j(t_i)$?

The answer for the recovery proceeds is straightforward. Hull and White have supposed constant or deterministic interest rates so that $E[RC_j(t_i)] = RC_j(t_i)$[6]. However in this paper interest rates are stochastic. The above equality is still valid when $C_j(t_i)$ is worth the face value which is a constant. But it is not anymore valid when $C_j(t_i)$ is worth the equivalent government bond value $G_j(t_i)$ which is a random variable, and letting $E[RG_j(t_i)] = RG_j(t_i)$ would not make sense. However wondering whether this expectation is equal to the forward value is common issue in finance and this is done as part of the second answer below.

The answer to whether $E[G_j(t_i)]$ is equal or not to $F_j(t_i)$ is positive. Indeed when set under a risk-neutral probability whose existence is ensured by the assumption of viable markets, the discounted prices of non-paying cash flows assets are martingales. As the discount factors are known values, this is equivalent to state that the expected prices of such assets are equal to their forward prices. As a check, a specific proof is given in Annex 1.

To summarize, in the case of zero-coupon bonds we get the Hull-White pricing formula but <u>adjusted</u> for stochastic interest rates:

Case 1: recovery of face value

$$G_j - B_j = \sum_{i=1}^{i=j} p_i \, \upsilon(t_i) \, (F_j(t_i) - R)$$

or (9)

Case 2: recovery of government equivalent bond

$$G_j - B_j = \sum_{i=1}^{i=j} p_i \, \upsilon(t_i) \, (F_j(t_i) - RF_j(t_i)) = \sum_{i=1}^{i=j} p_i \, \upsilon(t_i) \, F_j(t_i) \, (1-R)$$

In the second case, the original Hull-White formula features $RG_j(t_i)$ instead of $RF_j(t_i)$ which makes a significant difference.

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[6] Hull and White write they will relax the constant interest rates hypothesis further in their paper but we have not seen it.

III.1.2 The Coupon Bonds Case

The previous result can be extended to the case of a corporate bond paying continuously a coupon at rate q. Indeed applying the martingale property to the capitalized bond gives

$$\begin{aligned} B_j &= \upsilon(t_j)\ e^{qt_j}\ E[B_j(t_j)] = \sum_{i=1}^{i=j} p_i\ \upsilon(t_i)\ e^{qt_i}\ E[RC_j(t_i)] + \Big(1-\sum_{i=1}^{i=j} p_i\Big)\ e^{qt_j}\ \upsilon(t_j) \\ G_j &= \upsilon(t_j)\ e^{qt_j}\ E[G_j(t_j)] = \sum_{i=1}^{i=j} p_i\ \upsilon(t_i)\ e^{qt_i}\ E[G_j(t_i)] + \Big(1-\sum_{i=1}^{i=j} p_i\Big)\ e^{qt_j}\ \upsilon(t_j) \end{aligned} \qquad (10)$$

and the issue becomes: Do we have $e^{qt_i} E[G_j(t_i)] = F_j(t_i)$?

The answer is still yes as $E[G_j(t_i)] = \upsilon^{-1}(t_i)\ e^{-qt_i}\ G_j$ from the above martingale first equality and as $\upsilon^{-1}(t_i)\ e^{-qt_i}$ is the relative cost of carry net of the revenues, i.e. $F_j(t_i) \equiv \upsilon^{-1}(t_i)\ e^{-qt_i}\ G_j$.

Note however that in this case the Hull-White pricing formula needs be adjusted with term e^{qt_i}:

Case 1: recovery of face value

$$G_i - B_i = \sum_{j=1}^{j=i} p_j\ \upsilon(t_j)\ (F_i(t_j) - e^{qt_i} R)$$

or (11)

Case 2: recovery of government equivalent bond

$$G_i - B_i = \sum_{j=1}^{j=i} p_j\ \upsilon(t_j)\ (F_i(t_j) - RF_i(t_j)) = \sum_{j=1}^{j=i} p_j\ \upsilon(t_j)\ F_i(t_j)\ (1-R)$$

But extension is not straightforward when the bond serves one shot coupons at some discrete times as most benchmark straights do in actual markets, as we shall see now.

The martingale property does not apply to a stream of coupons plus principal if they are all consumed. But if they are all capitalized then the stream becomes an asset that does not pour out any cash flow, and consequently its discounted price becomes a martingale.

Now there are infinite ways of capitalizing coupons but all of them have the same present value B_j for simple non-arbitrage reason. Reinvesting the coupons in zero-coupon government bonds may do it but raises the question of using spot or forward bonds. The spot choice leads as lending forward the uncertain coupons from the corporate bond brings in a lot of complexity.

Let us consider a three period example from which it will be easy to generalize the results found to any number of periods.

The corporate bond B is scheduled to pay three coupons worth \$c at dates t_1, t_2, and t_3 and to redeem the principal worth \$1 at date t_3. The t_is are also the default times. When default occurs, the bondholders retrieve $RC(t_i)$. The equivalent government bond G pays safely the same coupons at the same dates. At a coupon date t_i, $B(t_i)$, $G(t_i)$ and $C(t_i)$ are computed with the just paid coupon, contrary to market practice.

Coupons are reinvested in one-period zero-coupon government bonds at future spot prices $\upsilon_i(t_i,t_{i+1})$, which are random variables at the current time.

In the presence of coupons we still get the adjusted Hull-White pricing formula for the three-period bond:

Case 1: recovery of face value

$$G_3 - B_3 = \sum_{j=1}^{j=3} p_j \, \upsilon(t_j) \, (F_3(t_j) - R(1+c))$$

or (12)

Case 2: recovery of government equivalent bond

$$G_3 - B_3 = \sum_{j=1}^{j=3} p_j \, \upsilon(t_j) \, F_3(t_j) \, (1-R)$$

Proof is given in Annex 2.

Extension to any-period bonds is obvious although it requires handling many indexed sums and products. Proof is shortened if it admits that the forward price of any fixed income instrument is equal to its expected value under a risk-neutral probability whose existence is ensured by the viable markets hypothesis.

At this point the adjusted Hull-White formula looks fairly robust. However it should not hide some traps set to the market practitioners.

Indeed in its setting all spot prices are dirty prices. On a pay-down day these prices must include the full coupon (whereas in practice they do on the day before). As a consequence the cash flows paid exactly at maturity by the bond must not be stripped off the forward prices.

Second, for $C(t_i)$ the Hull-White article asks to use $G(t_i)$ instead of $F(t_i)$: (page 8) ''In what follows we will consider two assumptions about the claim amount. The first is that it equals the no-default value of the bond at the time of the default;''. This is clearly wrong as shown in this paper. However a textbook[7] by Hull displays numerical results from which the correct use of forward prices can be inferred.

In sum, the much-praised distinguishing feature of feeding with coupon bearing bonds the Hull-White engine needs to put the hands in the dirty oil, otherwise the risk of mispricing is high.

III.2 The Jarrow-Turnbull Model Needs not a Lattice

The above formulae in II.2 look very cumbersome when maturity goes up but fortunately many discount factors collapse into default-free zero-bond prices. In this paper we propose the following formula to faster infer step by step the λ's:

$$\upsilon(0,T) = B(0,T)\ \{\ \delta\lambda(0) + \delta \sum_{j=1}^{j=T-1} \lambda(j)\ [\ \prod_{k=0}^{k=j-1} (1-\lambda(k))\ ] + \prod_{k=0}^{k=T-1} (1-\lambda(k))\ \} \qquad (13)$$

where B(0,T) is the government zero-coupon obligation maturing at T+1.

Knowing the λ's from λ(0) to λ(T−2), and the two bond prices υ(0,T) and B(0,T), allows to infer λ(T−1) easily for the formula is linear in λ(T−1).

Another way to present the bootstrap of the λ's is:

$$\lambda(0) = \frac{1-M(0)}{1-\delta} \text{ with } M(0) = \frac{\nu(0,1)}{B(0,1)} \qquad \lambda(1) = \frac{1-M(1)}{1-\delta} \text{ with } M(1) = \cfrac{\cfrac{\nu(0,2)}{B(0,2)}-\lambda(0)\delta}{1-\lambda(0)}$$

$$\lambda(2) = \frac{1-M(2)}{1-\delta} \text{ with } M(2) = \cfrac{\cfrac{\cfrac{\nu(0,3)}{B(0,3)}-\lambda(0)\delta}{1-\lambda(0)}-\lambda(1)\delta}{1-\lambda(1)}$$

$$\lambda(T-1) = \frac{1-M(T-1)}{1-\delta} \text{ with } M(T-1) = \cfrac{\cfrac{\begin{matrix}\cfrac{\cfrac{\cfrac{\cfrac{\nu(0,T)}{B(0,T)}-\lambda(0)\delta}{1-\lambda(0)}-\lambda(1)\delta}{1-\lambda(1)}-\lambda(2)\delta}{1-\lambda(2)}\cdots\cdots \\ \ddots\end{matrix}}{1-\lambda(T-3)}-\lambda(T-2)\delta}{1-\lambda(T-2)} \qquad (14)$$

These recursive structures get rid of the lattice, rely only on two term structures — government and corporate with same credit risk — and thus witness a calculation complexity comparable to Hull-White's.

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[7] ''Options, futures, & other derivatives'', chapter 26, John C. Hull, 5th international edition, Prentice Hall finance series, 2003.

III.3 The Jarrow-Turnbull Model Handles Coupon Bonds

To see that Jarrow-Turnbull allows to extract implied probabilities of default from corporate coupon bonds let us analyze it in the two-period case, the one-period being a case where coupon bonds are priced as pure discounts. Suppose the first probability of default, $\lambda(0)$, is known and the corporate bond pays interest c on a 1 dollar principal. To parallel the adjusted Hull-White model, assume that when the obligor defaults it services the debt by only a fraction δ of the remaining coupons at their payment dates and redeems only a fraction δ of the principal at maturity. The lattice becomes:

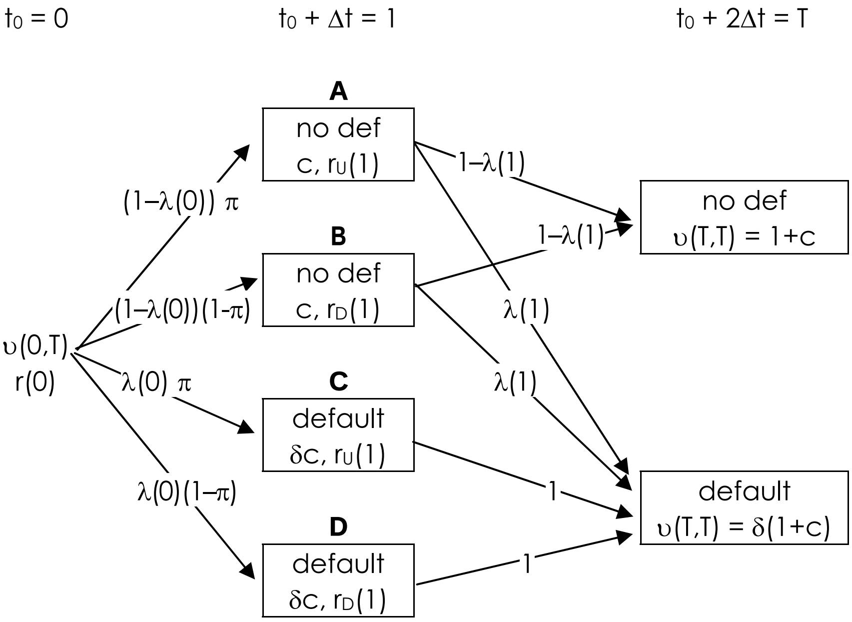

Diagram 2

and the calculations follow suit, with prices υ net of just paid full coupons:

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(15)\quad \begin{cases} \upsilon_A(1,2) = e^{-r_U(1)}\,[1-\lambda(1)+\lambda(1)\,\delta(1+c)] \qquad \upsilon_B(1,2) = e^{-r_D(1)}\,[1-\lambda(1)+\lambda(1)\,\delta(1+c)] \\ \upsilon_C(1,2) = e^{-r_U(1)}\,\delta(1+c) \qquad \upsilon_D(1,2) = e^{-r_D(1)}\,\delta(1+c) \end{cases}
$$

$$
\begin{aligned}
\upsilon(0,2) &= e^{-r(0)}\,\{\,[1-\lambda(0)]\,\pi\,[\upsilon_A(1,2)+c] + [1-\lambda(0)]\,(1-\pi)\,[\upsilon_B(1,2)+c] + \lambda(0)\,\pi\,[\upsilon_C(1,2)+\delta c] + \lambda(0)\,(1-\pi)\,[\upsilon_D(1,2)+\delta c]\,\} \\
&= e^{-r(0)}\,\{\,[1-\lambda(0)]\,\pi\,[e^{-r_U(1)}\,[1-\lambda(1)+\lambda(1)\,\delta]\,(1+c)+c] \\
&\qquad + [1-\lambda(0)]\,(1-\pi)\,[e^{-r_D(1)}\,[1-\lambda(1)+\lambda(1)\,\delta]\,(1+c)+c] \\
&\qquad + \lambda(0)\,\pi\,[e^{-r_U(1)}\,\delta(1+c)+\delta c] + \lambda(0)\,(1-\pi)\,[e^{-r_D(1)}\,\delta(1+c)+\delta c]\,\} \\
&= e^{-r(0)}\,\{\,(1-\pi+\pi)\,[1-\lambda(0)]\,c \\
&\qquad + [(1-\pi)\,e^{-r_D(1)} + \pi\,e^{-r_U(1)}]\,[1-\lambda(0)]\,[1-\lambda(1)+\lambda(1)\,\delta]\,(1+c) \\
&\qquad + [(1-\pi)\,e^{-r_D(1)} + \pi\,e^{-r_U(1)}]\,\lambda(0)\,\delta(1+c) + [(1-\pi)\,\lambda(0)+\lambda(0)\,\pi]\,\delta c\,\}
\end{aligned}
$$

$$= B(0,1)\ \{\ [1-\lambda(0)]\ c + B(1,2)[1-\lambda(0)]\ [1 - \lambda(1) + \lambda(1)\ \delta]\ (1+c)$$

$$+ B(1,2)\ \lambda(0)\ \delta(1+c) + \lambda(0)\ \delta c\ \}$$

Finally:

$$\upsilon(0,2) = B(0,1)\ \{\ \lambda(0)\ \delta c + [1-\lambda(0)]\ c\ \}$$

$$+ B(0,2)\ \{\ \lambda(0)\ \delta(1+c) + [1-\lambda(0)]\ [1 - \lambda(1)]\ (1+c) + [1-\lambda(0)]\ \lambda(1)\ \delta(1+c)]\ \}$$

This is the same type of equation as the original Jarrow-Turnbull formula but that takes account of the corporate bond fixed coupon c. It is linear in λ(1) and depends on known λ(0) and bond prices, so that extracting λ(1) is straightforward.

A general bootstrapping formula, linear in λ(T-1), can be derived:

$$\begin{gathered}
\upsilon(0,T) = \sum_{i=1}^{i=T} B(0,i)\ E[\text{coupon}_i] + B(0,T)\ E[\text{principal payment}] \\
\text{with} \\
E[\text{principal payment}] = \lambda(0)\ \delta + \sum_{j=1}^{j=T-1} [\ \lambda(j) \prod_{k=0}^{k=j-1} (1-\lambda(k))\ ]\ \delta + \prod_{k=0}^{k=T-1} (1-\lambda(k)) \\
\text{and} \qquad\qquad\qquad\qquad\qquad\qquad (16) \\
E[\text{coupon}_i] = \{\ \lambda(0)\ \delta + \sum_{j=1}^{j=i-1} [\ \lambda(j) \prod_{k=0}^{k=j-1} (1-\lambda(k))\ ]\ \delta + \prod_{k=0}^{k=i-1} (1-\lambda(k))\ \}\ c
\end{gathered}$$

To conclude there is no impediment for Jarrow-Turnbull to accept fixed-interest bearing credit-risky bonds.

III.4 Hull-White and Jarrow-Turnbull: Same Credit Engine

The previous sections have shown that the Hull-White model has in fact no advantageous feature over the Jarrow-Turnbull. Now if the two models don't provide the same default probabilities one may still be preferred on belief that it is closer to the true values, for after all both only approximate them.

This section shows that when the two models share a same basic setting then they yield the same probabilities.

III.4.1 The Zero-Coupon Bonds Case

The common framework consists of zero-coupon bonds, either government or corporate, and a constant recovery rate. Interest rates need not be deterministic.

A fundamental assumption to allow perfect reconciliation is the choice of the recovery basis in Hull-White, which proposes either the forward price of the equivalent bond or the face value plus accruals. We choose the first option in order to be consistent with the Jarrow Turnbull engine which postpones till maturity a recovery applied to the principal: this amounts to apply recovery earlier to the discounted value of the principal, which is equal to the forward price of a government bond in the underlying risk-neutral world.

Let us start with the one-period case ($T = t_1$) pricing equations:

$$(17)\quad \begin{cases} \text{Hull-White} & G_1 - B_1 = p_1\, \upsilon(t_1)\,[\,F_1(t_1) - R\,C_1(t_1)\,] \\ \text{Jarrow-Turnbull} & \upsilon(0,1) = e^{-r(0)}\,[\,1 - \lambda(0) + \delta\,\lambda(0)\,] \end{cases}$$

Now notice that $\upsilon(t_1) = G_1$, that at maturity $F_1(t_1) = C_1(t_1) = 1$, that $\delta = R$, and let us use the Hull-White notation G_1, B_1 and R:

$$(17')\quad \begin{cases} \text{Hull-White} & G_1 - B_1 = p_1\, G_1\,(\,1 - R\,) \\ \text{Jarrow-Turnbull} & B_1 = G_1\,[\,1 - \lambda(0) + R\,\lambda(0)\,] = G_1\,[\,1 - (1 - R)\,\lambda(0)\,] \end{cases}$$

Simple calculus brings $p_1 = \lambda(0) = \frac{G_1 - B_1}{B_1}\frac{1}{1-R}$. (17")

Therefore, in the one-period case, the two credit engines give the same default probability.

Now turn to the two-period case ($T = t_2$) pricing equations:

$$(18)\quad \begin{cases} \text{Hull-White} & G_2 - B_2 = p_1 G_1[\,F_2(t_1) - R\,C_2(t_1)\,] + p_2 G_2[\,F_2(t_2) - R\,C_2(t_2)\,] \\ \text{Jarrow-Turnbull} & \upsilon(0,2) = B(0,2)\,\{\,\lambda(0)\,\delta + [1-\lambda(0)]\,[1 - \lambda(1) + \delta\,\lambda(1)]\,\} \end{cases}$$

For consistency with Jarrow-Turnbull $C_2(t_1)$ is chosen to be $F_2(t_1)$, and in the absence of coupon $F_2(t_1) = e^{r(0)}\,G_2 = G_2 / G_1$. Again we use the same notations borrowed form Hull-White as in the one-period:

$$(18')\quad \begin{cases} \text{Hull-White} & G_2 - B_2 = p_1 G_1[G_2 / G_1 - R\,G_2 / G_1\,] + p_2 G_2(\,1 - R\,) \\ \text{Jarrow-Turnbull} & B_2 = G_2\,\{\,\lambda(0)\,R + 1 - \lambda(0) + [1-\lambda(0)]\,\lambda(1)\,(R - 1)\,\} \end{cases}$$

The system boils down to:

$$(18")\quad \begin{cases} \text{Hull-White} & \frac{G_2 - B_2}{G_2}\frac{1}{1-R} = p_1 + p_2 \\ \text{Jarrow-Turnbull} & \frac{G_2 - B_2}{G_2}\frac{1}{1-R} = \lambda(0) + [1-\lambda(0)]\,\lambda(1) \end{cases}$$

As we have got previously that $p_1 = \lambda(0)$ we now have $p_2 = [1-\lambda(0)]\,\lambda(1)$. (18"')

That $p_2 \neq \lambda(1)$ is not surprising as the p's are unconditional and the λ's conditional probabilities. The relation found between p_2, $\lambda(0)$ and $\lambda(1)$ fortunately respects the Bayes formula:

$$P(\text{default in t =2} \mid \text{no default in t=1}) = \frac{P(\text{default in } t=2 \text{ and no default in } t=1)}{P(\text{no default in } t=1)}$$

$$\lambda(1) = \frac{p_2}{1-\lambda(0)}$$

Therefore, in the two-period case, the two credit engines give the same default probabilities.

A recursive proof given in Annex 3 generalizes this result to n-period.

As a conclusion the two credit engines deliver the same implied default probabilities, otherwise saying, they are identical.

III.4.2 The Coupon Bonds Case

Incorporating coupons in the previous proof is tantamount to several pages of messy technical writings. A numerical simulation based on the example presented by Hull in his textbook will dramatically alleviate the burden.

Below are data about bonds issued by a same corporation:

Hull White Model		*Table 1*	Implied Probabilities	
Maturity	Coupon Rate	Yield to Maturity	Claim = No Default Forward Value	Claim = Face Value + Accrued Interest
1	7%	6.60%	**0.0214**	0.0214
2	7%	6.70%	**0.0238**	0.0236
3	7%	6.80%	**0.0261**	0.0258
4	7%	6.90%	**0.0284**	0.0278
5	7%	7.00%	**0.0306**	0.0299

Table 1

Probabilities are slightly different from the original because here the coupons are annual versus semi-annual in the textbook, with the corresponding compoundings. The probabilities are computed with paragraph III.1.2 adjusted Hull White model, a Treasury yield curve flat at 5%, and a recovery rate of 30%. They are unconditional.

As in the previous paragraph, the basic setting under study retains the forward price for the claim. Thanks to section III.3 formula, The Jarrow Turnbull model is able to provide the conditional default probabilities. Then Bayes formula translates them into unconditional numbers.

Jarrow Turnbull Model		*Table 2*	Implied Probabilities	
			Conditional	Unconditional
Maturity	Coupon Rate	Yield to Maturity	Claim = No Default Forward Value	
1	7%	6.60%	0.0214	**0.0214**
2	7%	6.70%	0.0243	**0.0238**
3	7%	6.80%	0.0274	**0.0261**
4	7%	6.90%	0.0306	**0.0284**
5	7%	7.00%	0.0340	**0.0306**

Table 2

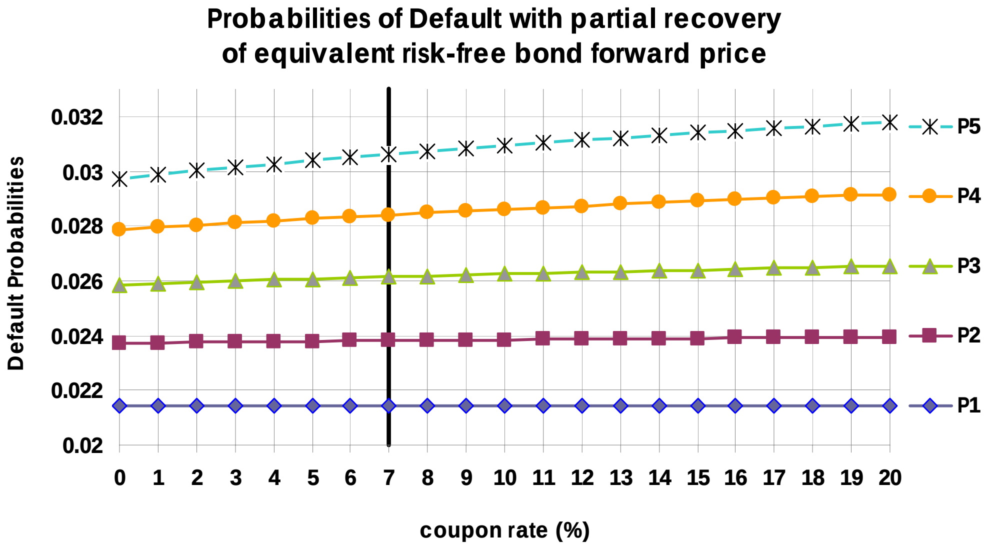
The default probabilities of the two models are thus identical. The same result has been reached with coupon rates ranging from 0 to 20%. Hence we can conclude, without formal (and boring) proof but with strong simulation evidence, that the two credit engines are again the same.

IV) OTHER PRICING CONSIDERATIONS

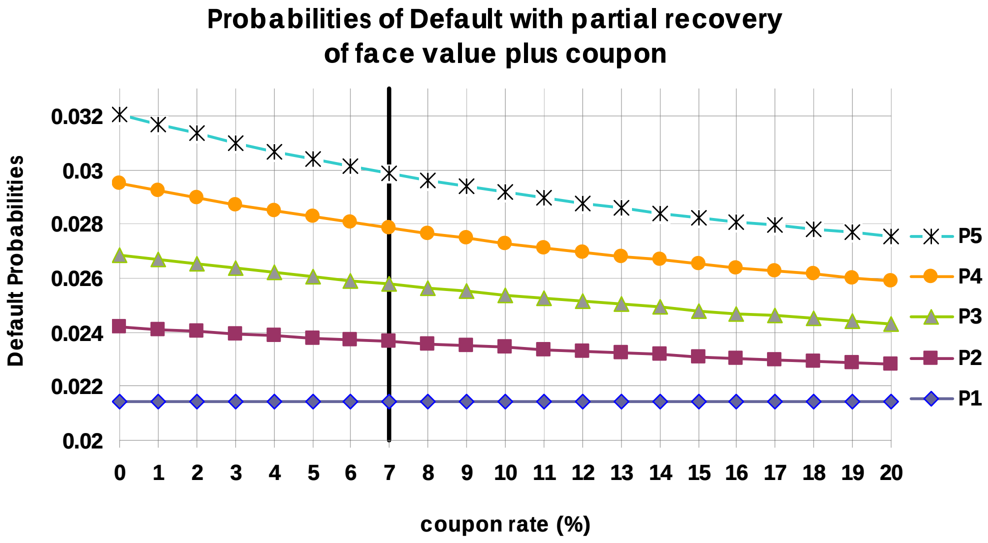
This section points to some pitfalls in pricing credit-risky bonds and explains how to use properly the previous model.

IV.1 Pricing with a Credit-Risky Yield Curve

Going back to Hull's textbook example, using the yield curve to price the corporate bonds leads to the following undesirable feature: the default probabilities depend on the coupon level.



Graph1



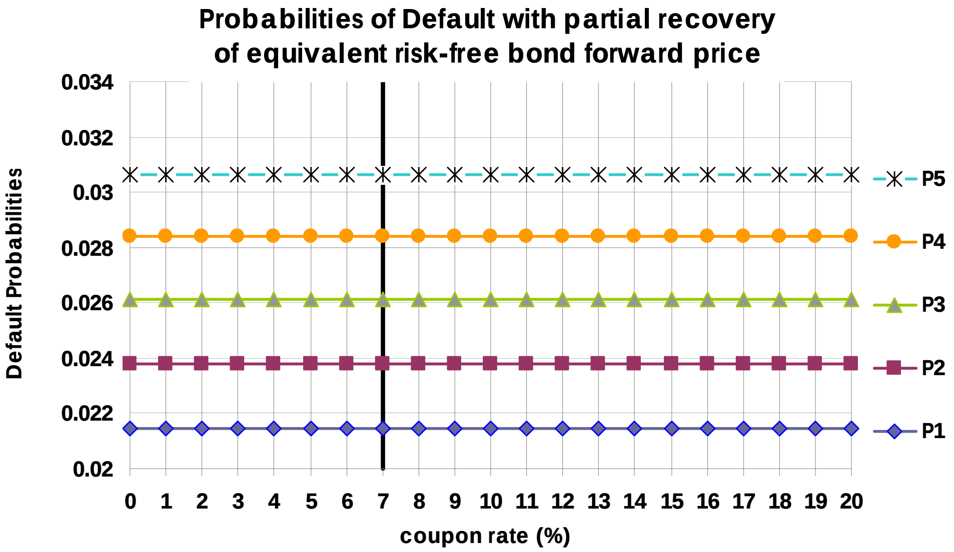
Graph 2

Yet everything in the model induces to state that the probabilities must be independent of the coupon and remain constant at the 7% values. But there is even worse: the dependency is very much changing according to the case of recovery claim, as shown in Graphs 1 and 2.

IV.2 Pricing with a Credit-Risky Zero-Coupon Curve

The yield curve approach to price bonds, whether credit-risky or not, has long been criticized and is today almost always replaced with the zero-coupon curve approach. But does that one deserve its success in the presence of credit risk ?

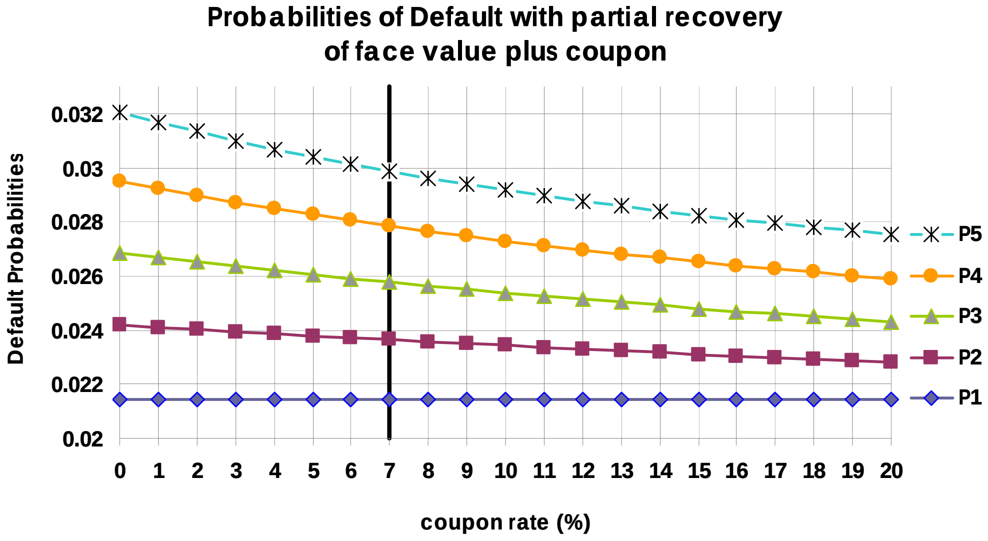
To answer the question we turn again to Hull's textbook example and classically bootstrap the zero-coupon rates from the yields to maturity. Then we discount back all the cash flows from any corporate bond with the corresponding maturity zero-rates. We price that way 21 sets of five bonds identical to those of the example excepted that coupon level varies from 0% to 20%. Last we extract the implied probabilities of default for each set using the adjusted Hull White model with a claim equal either to the forward value or the face value plus accruals. The two following graphs gather our findings.



Graph 3

When the claim concerns the forward price of the equivalent bond the zero-rate curve pricing is consistent with the implied default probabilities, it doesn't distort them as Graph 3 witnesses. This finding is of a great utility for the practitioners, as they can avoid calculating the probabilities. They can use directly the corporate zero curves to discount back the bonds and get their prices. A formal proof needs be written to fully warrant this handy shortcut.

However when the claim stands on the face value plus accruals pricing with the zeroes does not restore the starting point probabilities.



Graph 4

Graph 4 shows that their values at 7% are violated at all coupon level. Therefore in this case the pricing with the corporate zeroes is not consistent with the underlying default probabilities of the issuing corporation.

IV.3 Pricing with Default Probabilities

In the light of what has been displayed one would strongly advise to impose the same default probabilities on all the bonds of a same credit risk. Indeed most claims will make the zero-curve common practice inconsistent with these probabilities.

We suggest to set up a panel of the most liquid bonds (benchmarks) and use either the Jarrow Turnbull or the adjusted Hull White model to derive the default probabilities that they convey. These numbers must not be changed further on in the pricing process. Both models give the same result.

Then the chosen model is invoked the way around in order to price any other bond of the same credit risk. It now uses the probabilities extracted from the benchmarks as inputs, along with the risk-free rates, recovery rates, and so on.

V) CONCLUSIONS

In this paper we have shown formally and numerically that the commonly opposed models from Jarrow and Turnbull and from Hull and White are in fact identical. We have extended the Jarrow Turnbull engine to accept coupon bonds, and simplified its use by getting rid of its trees. A thorough proof of the Hull White formula is given, which was missing in the founding article and textbook. This reveals that the formula relies extensively on a strong assumption, which is that the markets are viable, and that the formula needs be adjusted with regards to its original version. These two models are a major breakthrough in finance because they show that the usual way to price corporate bonds, i.e. resorting to a zero-coupon curve bootstrapped from market benchmarks, is inconsistent. Indeed it implies different default probabilities for different coupons. These issues are still of current events even if there is no new business in the credit markets. Indeed the outstanding positions are tremendous and require careful hedging. Choosing the right model and fully understanding it remains more than ever a competitive advantage.

VI) ANNEX 1

Here we show that $E[G_j(t_i)]$ is equal to $F_j(t_i)$. Indeed 1) $G_j = \upsilon(t_j)$ in the absence of coupon, 2) the classical forward rates compounding ensures that $\upsilon(t_j) = \upsilon(t_i)\,\upsilon(t_i,t_j)$, and 3) applying the martingale property yields $G_j = \upsilon(t_i)\,E[G_j(t_i)]$. Hence $\upsilon(t_i)\,E[G_j(t_i)] = \upsilon(t_i)\,\upsilon(t_i,t_j)$ i.e. $E[G_j(t_i)] = \upsilon(t_i,t_j)$. Now $\upsilon(t_i,t_j)$ is nothing but $F_j(t_i)$ since by definition $F_j(t_i) \equiv \upsilon^{-1}(t_i)\,G_j = \upsilon^{-1}(t_i)\,\upsilon(t_j) \equiv \upsilon(t_i,t_j)$.

VII) ANNEX 2

Following the method sketched in paragraph III.1.1 we have:

$$
\begin{aligned}
B &= E[\upsilon(t_3) B(t_3)] \\
&= E[\upsilon(t_3)\ RC(t_1)\ \upsilon_1(t_1,t_3)^{-1}]\ p_1 \\
&+ E[\upsilon(t_3)\ RC(t_2)\ \upsilon_2(t_2,t_3)^{-1} + \upsilon(t_3)\ c\ \upsilon_1(t_1,t_2)^{-1}\ \upsilon_2(t_2,t_3)^{-1}]\ p_2 \\
&+ E[\upsilon(t_3)\ RC(t_3) + \upsilon(t_3)\ c\ \upsilon_1(t_1,t_2)^{-1}\ \upsilon_2(t_2,t_3)^{-1} + \upsilon(t_3)\ c\ \upsilon_2(t_2,t_3)^{-1}]\ p_3 \\
&+ E[\upsilon(t_3)\ (1+c) + \upsilon(t_3)\ c\ \upsilon_1(t_1,t_2)^{-1}\ \upsilon_2(t_2,t_3)^{-1} + \upsilon(t_3)\ c\ \upsilon_2(t_2,t_3)^{-1}]\ p_4 \\
&= E[\upsilon(t_1)\ RC(t_1)]\ p_1 \qquad (19) \\
&+ E[\upsilon(t_2)\ RC(t_2)]\ p_2 + E[\upsilon(t_1)\ c]\ p_2 \\
&+ E[\upsilon(t_3)\ RC(t_3)]\ p_3 + E[\upsilon(t_1)\ c]\ p_3 + E[\upsilon(t_2)\ c]\ p_3 \\
&+ E[\upsilon(t_3)\ (1+c)]\ p_4 + E[\upsilon(t_1)\ c]\ p_4 + E[\upsilon(t_2)\ c]\ p_4
\end{aligned}
$$

with $p_4 = 1-p_1-p_2-p_3$

Note that to get this result we have applied twice the martingale property to the asset ''\$1 lent from t_2 to t_3'':

$$\text{Current price of asset} = E[\upsilon(t_2)\ 1] = E[\upsilon(t_3)\ \upsilon_2(t_2,t_3)^{-1}]$$

and to the asset ''\$1 lent from t_1 to t_3'':

$$\text{Current price of asset} = E[\upsilon(t_1)\ 1] = E[\upsilon(t_3)\ \upsilon_1(t_1,t_2)^{-1}\ \upsilon_2(t_2,t_3)^{-1}]$$

Now as in the previous paragraph notice that

$$G = \left(\sum_{i=1}^{i=3} p_i + 1 - \sum_{i=1}^{i=3} p_i\right) G$$

and use the martingale property again:

$$\begin{aligned} G &= E[\upsilon(t_1)\ G(t_1)] \\ &= E[\upsilon(t_2)\ (G(t_2) + c\upsilon_1(t_1,t_2)^{-1})] \\ &= E[\upsilon(t_3)\ (1+c + c\upsilon_1(t_1,t_2)^{-1}\ \upsilon_2(t_2,t_3)^{-1} + c\upsilon_2(t_2,t_3)^{-1})] \end{aligned} \tag{20}$$

then G can be rewritten as:

$$\begin{aligned} G = {} & E[\upsilon(t_1)\ G(t_1)]\ p_1 \\ & + E[\upsilon(t_2)\ G(t_2)]\ p_2 + E[\upsilon(t_2) c\upsilon_1(t_1,t_2)^{-1})]\ p_2 \\ & + E[\upsilon(t_3)\ (1+c)]\ p_3 + E[\upsilon(t_3) c\upsilon_1(t_1,t_2)^{-1}\upsilon_2(t_2,t_3)^{-1}]\ p_3 + E[\upsilon(t_3) c\upsilon_2(t_2,t_3)^{-1})]\ p_3 \\ & + E[\upsilon(t_3)(1+c)]\ p_4 + E[\upsilon(t_3) c\upsilon_1(t_1,t_2)^{-1}\upsilon_2(t_2,t_3)^{-1}]\ p_4 + E[\upsilon(t_3) c\upsilon_2(t_2,t_3)^{-1})]\ p_4 \\ = {} & E[\upsilon(t_1)\ G(t_1)]\ p_1 \\ & + E[\upsilon(t_2)\ G(t_2)]\ p_2 + E[\upsilon(t_1)\ c]\ p_2 \\ & + E[\upsilon(t_3)\ (1+c)]\ p_3 + E[\upsilon(t_1)\ c]\ p_3 + E[\upsilon(t_2)\ c]\ p_3 \\ & + E[\upsilon(t_3)\ (1+c)]\ p_4 + E[\upsilon(t_1)\ c]\ p_4 + E[\upsilon(t_2)\ c]\ p_4 \end{aligned} \tag{21}$$

with $p_4 = 1 - p_1 - p_2 - p_3$

Collecting all terms in B and G we find: (22)

$$G - B = E[\upsilon(t_1)\ (G(t_1) - RC(t_1))]\ p_1 + E[\upsilon(t_2)\ (G(t_2) - RC(t_2))]\ p_2 + E[\upsilon(t_3)\ (1 - RC(t_3))]\ p_3$$

Taking out the discount factors provides: (22')

$$G - B = \upsilon(t_1)(E[G(t_1)] - RE[C(t_1)])p_1 + \upsilon(t_2)(E[G(t_2)] - RE[C(t_2)])p_2 + \upsilon(t_3)(1 - RE[C(t_3)])p_3$$

To get the adjusted Hull White pricing formula one needs to check whether the expected price of the government bond is equal to its forward price. This equality is not ensured by the viable markets hypothesis as the bond pays off coupons.

According to the martingale equalities (20) we have:

$$\begin{aligned} E[G(t_1)] &= \upsilon(t_1)^{-1}\ G \\ E[G(t_2)] &= \upsilon(t_2)^{-1}\ G - c\ E[\upsilon_1(t_1,t_2)^{-1}] \\ E[G(t_3)] &= \upsilon(t_3)^{-1}\ G - c\ E[\upsilon_1(t_1,t_2)^{-1}\upsilon_2(t_2,t_3)^{-1}] - c\ E[\upsilon_2(t_2,t_3)^{-1}] \end{aligned} \tag{20'}$$

The forward prices of the government bond are by definition:

$F(t_1) = \upsilon(t_1)^{-1}\ G$

$F(t_2) = \upsilon(t_2)^{-1}\ G - c\ \upsilon(t_1,t_2)^{-1}$

$F(t_3) = \upsilon(t_3)^{-1}\ G - c\ \upsilon(t_1,t_3)^{-1} - c\ \upsilon(t_2,t_3)^{-1}$

where $\upsilon(t_i,t_{i+1})$ is the forward price of the zero-coupon one-period government bond. Note that the coupon at date t_i is included in the dirty price $G(t_i)$ and then must not show up in $F(t_i)$ computation.

As shown earlier (5):

$E[\upsilon(t_1)\ 1] = E[\upsilon(t_2)\ \upsilon_1(t_1,t_2)^{-1}]$

$E[\upsilon(t_2)\ 1] = E[\upsilon(t_3)\ \upsilon_2(t_2,t_3)^{-1}]$ (5')

$E[\upsilon(t_1)\ 1] = E[\upsilon(t_3)\ \upsilon_1(t_1,t_2)^{-1}\ \upsilon_2(t_2,t_3)^{-1}]$

Taking out the spot discount factors from the expectation operator yields:

$E[\upsilon_1(t_1,t_2)^{-1}] = \upsilon(t_2)^{-1}\ \upsilon(t_1)$

$E[\upsilon_2(t_2,t_3)^{-1}] = \upsilon(t_3)^{-1}\ \upsilon(t_2)$ (5")

$E[\upsilon(t_3)\ \upsilon_1(t_1,t_2)^{-1}\ \upsilon_2(t_2,t_3)^{-1}] = \upsilon(t_3)^{-1}\ \upsilon(t_1)$

The right hand side terms are nothing but the forward capitalization factors:

$\upsilon(t_2)^{-1}\ \upsilon(t_1) = \upsilon(t_1,t_2)^{-1}$

$\upsilon(t_3)^{-1}\ \upsilon(t_2) = \upsilon(t_2,t_3)^{-1}$

$\upsilon(t_3)^{-1}\ \upsilon(t_1) = \upsilon(t_1,t_3)^{-1}$

Hence $E[G(t_j)] = F(t_j)$ for $j = 1$, 2 and 3. In the special case of $j = 3$ we have $E[G(t_3)] = F(t_3) = 1$.

The pricing formula now becomes: (12')

$$G - B = \upsilon(t_1)(F(t_1) - RE[C(t_1)])\ p_1 + \upsilon(t_2)(F(t_2) - RE[C(t_2)])\ p_2 + \upsilon(t_3)(F(t_3) - RE[C(t_3)])\ p_3$$

VIII) ANNEX 3

Suppose that for $T > 1$ we have $p_T = [1-\lambda(0)]\ [1-\lambda(1)]\ldots[1-\lambda(T-2)]\ \lambda(T-1)$.

In the Hull-White engine, given that $F_{T+1}(t_i) = G_{T+1} / G_{T+1-i}$, the recursive pattern of $G_{T+1} - B_{T+1}$ unveiled in the two-period case is obvious and leads to:

$$\frac{G_{T+1}-B_{T+1}}{G_{T+1}}\frac{1}{1-R} = p_1 + p_2 + \ldots + p_{T+1} \qquad (23)$$

But it is very tedious to disentangle the Jarrow-Turnbull pricing formula to reach the following equality:

$$\frac{G_{T+1}-B_{T+1}}{G_{T+1}}\frac{1}{1-R} = \lambda(0) + [1-\lambda(0)]\lambda(1) + \dots + [1-\lambda(0)][1-\lambda(1)]\dots[1-\lambda(T-1)]\lambda(T) \qquad (24)$$

However noticing that by assumption all the paths to maturity are mutually independent, and that the interest rate lattice doesn't play any role as stressed in section III.2, it suffices to model a two-period credit lattice with T+1 independent paths:

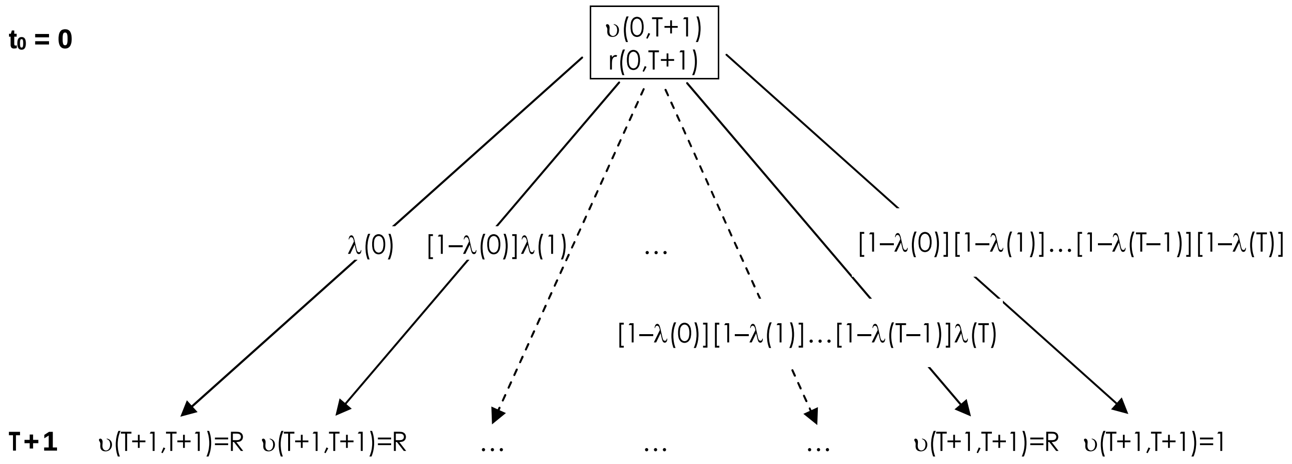


Diagram 3

The martingale equality between t = 0 and t = T+1 gives: $\frac{B_{T+1}}{G_0^{-1}} = E[\frac{B_{T+1}(T+1)}{G_{T+1}^{-1}}]$ and by developing the expectation: (23')

$$\frac{B_{T+1}}{G_{T+1}} = \{ \lambda(0) + [1-\lambda(0)]\lambda(1) + \dots + [1-\lambda(0)][1-\lambda(1)]\dots[1-\lambda(T-1)]\lambda(T) \} R + [1-\lambda(0)][1-\lambda(1)]\dots[1-\lambda(T-1)][1-\lambda(T)]$$

Besides the probabilities of the T+1 paths sum to unity:

$$1 = \lambda(0) + [1-\lambda(0)]\lambda(1) + \dots + [1-\lambda(0)][1-\lambda(1)]\dots[1-\lambda(T-1)]\lambda(T) + [1-\lambda(0)][1-\lambda(1)]\dots[1-\lambda(T-1)][1-\lambda(T)]$$

Subtracting the last two equations yields:

$$1 - \frac{B_{T+1}}{G_{T+1}} = \{ \lambda(0) + [1-\lambda(0)]\lambda(1) + \dots + [1-\lambda(0)][1-\lambda(1)]\dots[1-\lambda(T-1)]\lambda(T) \} (1-R) \qquad (23'')$$

and, after rearranging, the Jarrow-Turnbull formula looked after:

$$\frac{G_{T+1}-B_{T+1}}{G_{T+1}}\frac{1}{1-R} = \lambda(0) + [1-\lambda(0)]\lambda(1) + \dots + [1-\lambda(0)][1-\lambda(1)]\dots[1-\lambda(T-1)]\lambda(T) \qquad (23''')$$

When this formula is paralleled with Hull-White's

$$\frac{G_{T+1}-B_{T+1}}{G_{T+1}}\frac{1}{1-R} = p_1 + p_2 + \dots + p_{T+1}$$

and as $p_T = [1-\lambda(0)]\ [1-\lambda(1)]\dots[1-\lambda(T-2)]\ \lambda(T-1)$ it follows that:

$$p_{T+1} = [1-\lambda(0)]\ [1-\lambda(1)]\dots[1-\lambda(T-1)]\ \lambda(T) \qquad (18'''')$$

As a conclusion, if the two models hold then they share the same default probabilities.

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