A dynamic mean-field contagion model with default

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Extended abstract

To model rating transitions of firms, Lando and Skodeberg in [4] propose a continuous time Markov process, which fits real data better than a discrete time Markov chain. However they also point out the presence of non-Markovian patterns in rating histories. In particular they detect the presence of *momentum* in the data, i.e. firms which have just been downgraded have a greater probability of another downgrade. To take into account this behaviour, Christensen et al. in [1] propose a Hidden Markov Model for the rating evolution, introducing some hidden excited states which correspond to firms that have just been downgraded. Elliot and Korolkiewicz in [3] also use a Hidden Markov model for the rating migration, where the hidden Markov chain models the real transitions which are observed with noise.

A novel approach to rating migration can be found in [2]. The aim of that paper is to study the propagation of financial distress in a network of interacting firms facing credit risk, using particle systems. They associate to each firm i in the network two random variables, σ_i and ω_i, both taking values in $\{-1, 1\}$. The former, σ_i, is observable and it represents the rating of firm i, while the latter, ω_i, is not observable and it is an indicator of the financial health of firm i. Therefore, in their model, the rating indicator does not necessarily reflect the true financial condition of the firm. A good rating (resp. a good financial health) corresponds to $\sigma_i = 1$ (resp. $\omega_i = 1$). Even though the couple (σ_i, ω_i) is modelled as a Markov process, the rating indicator of a firm is not Markovian. In particular, the probability of a downgrade in the rating is higher if the financial health of the firm is bad.

We propose an extension of the model in [2], adding a default state, $(\sigma_i, \omega_i) = (0, 0)$, for a firm. We consider a network of N potential firms, and we assume that some of them are initially out of the market, that is their initial state is the default one. The number of firms which are alive at time t is $\bar{N}(t) \leq N$.

As in [2], we consider a mean-field interaction between the firms, that is each firm interacts with the other firms in the same way, without a geometry in the configuration space. We consider an interacting intensity model, where we have to specify the different intensity rates at which the transitions between two states take place.

We define the following three key quantities:

$$m_N^\sigma := \frac{1}{N}\sum_{i=1}^N \sigma_i, \qquad \mu := \frac{1}{N}\sum_{i=1}^N |\sigma_i|, \qquad \bar{m}_N^\sigma := \begin{cases} \frac{m_N^\sigma}{\mu} & \text{if } \mu \neq 0 \\ \bar{m}_0 & \text{if } \mu = 0, \end{cases}$$

for some $\bar{m}_0 \in [-1, 1]$. The variable $\mu \in [0, 1]$ is the percentage of alive firms, while the variable $\bar{m}_N^\sigma \in [-1, 1]$ represents the *global financial indicator* of the alive firms.

Let q_0 be the initial distribution of the model. It is a distribution on the set of possible configurations:

$$\{(1, 1), (1, -1), (-1, 1), (-1, -1), (0, 0)\}^N$$

As in [2], we rule out the possibility that two firms change their states at the same time. Then we define the transition intensities of a firm as follows

$$\begin{aligned} \sigma_i &\rightsquigarrow -\sigma_i \quad \text{with intensity} && e^{-\beta\sigma_i\omega_i}, \\ \omega_i &\rightsquigarrow -\omega_i \quad \text{with intensity} && e^{-\gamma\omega_i\bar{m}_N^\sigma}, \\ (\sigma_i, \omega_i) &\rightsquigarrow (0, 0) \quad \text{with intensity} && a_1 e^{-\gamma\left(\omega_i + a_2(\bar{m}_N^\sigma - a_3\mu)\right)}, \\ (0, 0) &\rightsquigarrow (\sigma_i, \omega_i) = (\pm 1, \pm 1) \quad \text{with intensity} && q(\sigma_i, \omega_i)e^{\gamma a_2(\bar{m}_N^\sigma - a_3\mu)}, \end{aligned}$$

where $\beta, \gamma > 0$, $a_1 \in (0, 1)$ and $0 \leq a_2, a_3 \leq 1$. The transitions between non-default states are exactly as in [2]. We can interpret β as an indicator of the efficiency of the rating system, while γ can be seen as an index of how much the external world influences the dynamic of the firms. Note that the transition intensity to the default state is decreasing in $\bar{m}_N^\sigma$, so that the default is less likely if the health of the market is good, and it is increasing in μ, so that the default is more likely if there are many firms in the market, i.e. if the competition in the market is high. The transition intensity from the default state to non-default states is positive. This allows to model the entry of new firms in the market: this is more likely when the global health is good and when there is less competition (i.e. μ is low)

We first simulate this model, finding interesting features related to the equilibrium levels of the quantities m_N^σ, $\bar{m}_N^\sigma$ and μ. In particular, we find how the choice of the parameters in the model qualitatively affects the long-run macroscopic behaviour of the model. We also study the time evolution of the percentage of firms with the right rating, i.e. $(\sigma, \omega) \in \{(1, 1), (-1, -1)\}$, in particular how this quantity depends on the parameter β. Finally, as in [2], we consider our model in a risk management perspective, analyzing the distribution of the portfolio losses when the number of firms N is large.

References

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