Value-at-Risk, limited liability and the moral-hazard problem

Vanda Tulli* Gerd Weinrich†

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Abstract

In this paper we suggest to employ Value-at-Risk as a means to induce a firm that seeks to maximize expected profit under risk of bankruptcy and limited liability to behave in a kind of risk-averse manner. It is shown that there exists a well-defined solution function which relates output to the firm's equity base, but that due to moral hazard this function is not monotone. In a dynamic framework equity becomes a choice variable and, while preserving limited liability, the VaR-constraint deters the firm from gambling, thus eliminating the moral-hazard problem.

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[*] *Corresponding author.* Dipartimento di Metodi Quantitativi per le Scienze Economiche e Aziendali, Università di Milano-Bicocca, Piazza dell'Ateneo Nuovo 1, 20126 Milano, Italy. Phone: +39-02-6448.3164, e-mail: vanda.tulli@unimib.it.

[†] Catholic University of Milan, Largo Gemelli 1, 20123 Milano, Italy. Phone: +39-02-7234.2728, http://docenti.unicatt.it/ita/Gerd_Hellmut_Weinrich, e-mail: gerd.weinrich@unicatt.it.

1 Introduction

Value-at-Risk is a concept intended to be a measure of the risk of financial investments. Although it is commonly used by investment banks to evaluate the market risk of the assets that they hold in their portfolio, it may also be employed by banks to evaluate the creditworthiness of projects submitted to them by firms and to determine the corresponding terms of credit. Vice versa, a bank may impose a Value-at-Risk constraint on a firm that asks for funding to limit its exposure to risk. The firm may then combine this with an objective - for example profit maximization - to determine its most preferred action. As a result it may show a kind of risk averse behaviour.

Specifically, since the return to an investment of a firm is inherently subject to uncertainty, to encourage a firm to run the corresponding risk the legal provision of limited liability has been introduced a long time ago. The drawback of that, which has become all the more obvious in the recent credit crunch crisis, is that the firm may become overconfident and act less carefully than it would if it were fully liable to the outcome. In other words, there is a problem of moral hazard. The VaR-constraint suggests itself to counteract this problem. The major result of this paper is that in fact, in a dynamic context, under certain natural conditions it does so, also without abolishing limited liability.

More formally, let X be a real-valued random variable with continuous and strictly increasing distribution function $F(x)$ on the non-empty set $F^{-1}(0,1)$. Then, for given *confidence level* $\alpha \in (0,1)$ we define $VaR_\alpha[X]$ as that real number for which

$$F(-VaR_\alpha[X]) = \alpha. \tag{1}$$

This means that the probability that a realization of X is smaller than or equal to $-VaR_\alpha[X]$ is α. Equivalently, the probability that a realization of $-X$ (the loss) is larger than or equal to $VaR_\alpha[X]$ is α.[1] If $F(x)$ is influenced by variables (y, a), where y is a choice variable and a a parameter, then X has distribution function $F(x, y, a)$ and becomes a random function $X(y, a)$. Therefore $VaR_\alpha[X(y, a)]$ is determined by $F(-VaR_\alpha[X(y, a)], y, a) = \alpha$, and if in addition the firm desires to maximize $EX(y, a)$ under the constraint that $VaR_\alpha[X(y, a)]$ be smaller or equal to some a priorily contemplated

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[1] This definition of VaR can be generalized to encompass the case of discontinuous and weakly increasing distribution functions $F(x)$ by setting $VaR_\alpha[X] = \inf\left(-F^{-1}[0,\alpha]\right)$. We do not use this here since it is not relevant for the model studied in this paper.

$v \in \mathbb{R}$, then its decision problem becomes

$$\begin{array}{c} \max_y EX(y,a) \\ s.t.\ VaR_\alpha[X(y,a)] \leq v \end{array}$$

with solution $\widehat{y}(a)$.

In this paper we study the nature of $\widehat{y}(a)$ in the specific economic context of a firm à la Greenwald-Stiglitz (1993). Then y is output and a is the equity base or retained capital that the firm can employ to finance its production. As has already been shown in Tulli and Weinrich (2009) for a static context, $\widehat{y}(a)$ shows a non-monotone behaviour. This is due to the moral-hazard effect of limited liability.

In a dynamic framework a becomes endogenous. By deciding how much of its profit realized in period t to distribute to shareholders and how much to keep within the firm for its operations in period $t+1$, a becomes a dynamic variable a_t for which an optimal path can be studied. It turns out that in the case that the subjective discount rate of shareholders is smaller than the inverse of the interest factor, i.e. $\beta < 1/(1+r)$, the firm will always choose a path along which the VaR-constraint is binding and, moreover, that the dynamically optimal sequence (a_t^*) is constant, i.e. $a_t^* = \overline{a}^* \geq 0$. In addition, if the elasticity of scale of the firm's technology is not too small, i.e. sufficiently close to constant returns, then $\overline{a}^* > 0$. As a consequence, also output $\widehat{y}(a_t^*) = \widehat{y}(\overline{a}^*)$ and expected dividends will be constant over time. Finally, and most importantly, the presence of the VaR-constraint will imply that the moral-hazard effect of limited liability does not bite. Vice versa, without a VaR-constraint in our model not only would there be moral hazard, but $\overline{a}^*$ would be zero.

Although Value-at-Risk has achieved high status because of being written into industry regulations (see e.g. Jorion (1997)), and is in fact a popular instrument used in practice (e.g. Bauer (2000), Pritsker (1997)), it also has been criticised for having weaknesses from the point of view of its axiomatic foundation as risk measure with desirable properties. This has led to refinements like, for instance, the Conditional Value-at-Risk (e.g. Lüthi and Doege (2005) and Rockafellar and Uryasev (2000)). Various aspects of the use of VaR and/or CVaR in optimization contexts have been discussed in Kast et. al. (1998), Rockafellar and Uryasev (2002) and Yiu (2004). The present paper's model, by using VaR as a constraint in profit maximization à la Greenwald-Stiglitz (1993), is elementary enough to not give rise to conceptional complications, on the one hand, and, nevertheless, appears distinctively novel and telling, on the other.[2]

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[2] For example, the problem usually related to VaR that it provides no handle on the

In section 2 we present the model and summarize the results of the static set-up. Section 3 represents the main contribution of the current paper as it extends the static model to a dynamic framework, and section 4 concludes. An appendix contains for completeness some technical results regarding the static model which, due to the assumption of zero bankruptcy cost, is here a simplified and shortened version of Tulli and Weinrich (2009).

2 The Static Model

2.1 Assumptions

Following Greenwald and Stiglitz (1993), we consider a firm that produces a single output y using labour n as the only input, with labour requirement function $n = \Phi(y)$. The corresponding labour cost, wn, with w the nominal wage, is covered by the firm partly with own funds or retained capital, denoted a, and partly with debt capital, b, which it has obtained as a loan from banks. Thus $b = wn - a$.[3] The firm sells its output on a competitive market, and thus the output price, p, is considered as not controlled by the firm but determined by the market. Moreover, the firm does not know with certainty the value of p when it contracts labour because there is a time lag in production, and the firm has to hire workers before the uncertainty is resolved. Denoting with $R = 1 + r$ the interest factor and r the interest rate, the resulting profit, according to Greenwald and Stiglitz (1993), is

$$\pi = py - R[w\Phi(y) - a] =: \pi(y, a, p). \tag{2}$$

Note that π is positive if the firm does not produce at all: in that case the firm can act as a lender of its capital a and earn Ra.

Let $g(p)$ denote a density describing the distribution of the random variable P, as believed by the firm. Then this induces, for the random variable $\Pi(y, a) := \pi(y, a, P)$, the density

$$f(y, a, \pi) = g(\Psi(y, a, \pi)) \frac{\partial \Psi(y, a, \pi)}{\partial \pi}, \tag{3}$$

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extent of the losses that might be suffered beyond the threshold amount will be overcome in our model by assuming - realistically, due to limited liability - a constant bankruptcy cost, i.e. independent of the size of the firm's chosen action.

[3] Greenwald and Stiglitz (1993) assume that, due to asymmetric information, firms cannot finance their production cost by issueing new equity. See e.g. Myers e Majluf (1984) for a formal justification of this argument.

where

$$\Psi(y,a,\pi) := \frac{\pi + R[w\Phi(y) - a]}{y} \tag{4}$$

is the inverse of the function $\pi(y,a,\cdot)$. Depending on the realization of P, profit may be negative. In that case the firm is not able to fully repay its debt b and we consider it to be bankrupt. Being an organization with limited liability, the firm then has to give up its assets and bear other possible costs which add up to a constant bankruptcy cost $c \geq 0$. The expected gain for a firm producing the output $y \geq 0$ being endowed with capital $a \geq 0$ is therefore

$$\int_{-\infty}^{0} -cf(y,a,\pi)d\pi + \int_{0}^{+\infty} \pi f(y,a,\pi)d\pi =: \ \Gamma(y,a) \tag{5}$$

If the firm were risk neutral and had no further constraint, it would maximize $\Gamma(y,a)$. If, however, the firm has to limit the probability of going bankrupt to $\alpha \geq 0$, then, setting

$$F(y,a) := \int_{-\infty}^{0} f(y,a,\pi)d\pi \ ,$$

its problem becomes

$$\begin{array}{c} \max_y \Gamma(y,a) \\ s.t. \ F(y,a) \leq \alpha \ . \end{array} \tag{6}$$

Recalling the definition of Value-at-Risk in (1), this is obviously equivalent to

$$\begin{array}{c} \max_y \Gamma(y,a) \\ s.t. \ VaR_\alpha\left[\Pi(y,a)\right] \leq 0. \end{array} \tag{7}$$

Thus we have obtained a formalization of the firm's decision problem involving VaR as an element to control its risk of bankruptcy. Note that the firm's degree of "risk aversion" (not risk aversion in the conventional sense) is expressed by the constant $\alpha \geq 0$. Any risk can be excluded by setting $\alpha = 0$. However, in that case the firm may be severely limited in its opportunities to realize a satisfactory profit. Therefore $\alpha > 0$ may be a better choice.[4]

Denote a solution to (7) by $\widehat{y}(a)$. To characterize it, we make the following assumptions which regard the price distribution, the technology and the bankruptcy cost:

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[4] In practice, the values most often employed by organizations are $\alpha = 0.01$ and $\alpha = 0.05$.

(A1) The price of the product is a random variable P with uniform distribution over its support $[0, 2]$.

(A2) $\Phi(y)$ is differentiable, strictly increasing and strictly convex. Moreover, $\Phi(0) = \Phi'(0) = 0$ and $\lim_{y\to\infty} \Phi'(y) = +\infty$.

(A3) The bankruptcy cost c is zero.

Although the above specification of the price distribution is the simplest to work with, it will not be easy to solve explicitly for $\widehat{y}(a)$; still, it will enable us to show the main point. Moreover, it is clear that there would be no problem to generalize the support to $[0, \overline{p}]$, $\overline{p} > 0$. The normalization to $\overline{p} = 2$ will simplify the calculations as $EP = 1$ but not be essential in any way to the results. Specifically (A1) implies

$$g(p) = \begin{cases} \frac{1}{2} & \text{if } 0 \le p \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and, recalling (3) and (4),

$$f(y, a, \pi) = \begin{cases} \frac{1}{2y} & \text{if } -R\left[w\Phi(y) - a\right] < \pi < 2y - R\left[w\Phi(y) - a\right] \\ 0 & \text{otherwise} \end{cases} . \tag{9}$$

From this we obtain

$$F(y, a) = \begin{cases} 0 & \text{if } 0 \le -R\left[w\Phi(y) - a\right] \\ \int_{-R[w\Phi(y)-a]}^{0} \frac{1}{2y} d\pi & \text{if } -R\left[w\Phi(y) - a\right] < 0 \le 2y - R\left[w\Phi(y) - a\right] \\ 1 & \text{if } 0 > 2y - R\left[w\Phi(y) - a\right] \end{cases}$$

$$= \begin{cases} 0 & \text{if } 0 \le -R\left[w\Phi(y) - a\right] \\ \frac{R}{2y}\left[w\Phi(y) - a\right] & \text{if } -R\left[w\Phi(y) - a\right] < 0 \le 2y - R\left[w\Phi(y) - a\right] \\ 1 & \text{if } 2y - R\left[w\Phi(y) - a\right] < 0 \end{cases} . \tag{10}$$

Regarding (A2), note that, since labour is the only input of production, strict convexity of $\Phi(y)$ as assumed in (A2) is equivalent to increasing marginal cost, which, moreover, tends to infinity as $\lim_{y\to\infty} \Phi'(y) = +\infty$.

(A3) is technically convenient without altering the substance of the results. The case $c \ge 0$ has been dealt with in Tulli and Weinrich (2008).

2.2 Solution of the static problem

Coming back to problem (7), denote a solution to $VaR_\alpha\left[\Pi(y,a)\right] = 0$ by $\widehat{y}_I(a)$. Under (A1) it is implicitly determined, according to (6) and (10), by

$$R\left[w\Phi(\widehat{y}_I(a)) - a\right] = 2\alpha\widehat{y}_I(a) \ . \tag{11}$$

As we show in the Appendix (Lemma 1), $\widehat{y}_I(a)$ is a strictly increasing function. Next set $\widehat{y}_{II}(a) := \arg\max_y \Gamma(y,a)$. To understand its behaviour, we first determine $\Gamma(y,a)$. To this end we define

$$\underline{y}(a) := \Phi^{-1}(\frac{a}{w}) \ . \tag{12}$$

By (A2) $\underline{y}(a)$ is a strictly increasing function with $\underline{y}(0) = 0$ and $\lim_{a\to\infty}\underline{y}(a) = \infty$. For any (y,a) such that $y \le \underline{y}(a)$, by (2) there is no risk of bankruptcy, and by (9) $\Gamma(y,a)$ coincides with

$$\mu\left(y,a\right) := \int_{-\infty}^{+\infty} \pi f(y,a,\pi)d\pi = E\Pi(y,a) = y - R[w\Phi(y) - a] \ . \tag{13}$$

Instead for (y,a) such that $y \ge \underline{y}(a)$ consider, using (9),

$$\mu_1\left(y,a\right) := -\int_{-\infty}^{0} \pi f(y,a,\pi)d\pi = -\frac{1}{4y}\left[\pi^2\right]_{-R[w\Phi(y)-a]}^{\overline{\pi}} \ , \tag{14}$$

where

$$\overline{\pi} := \min\left\{2y - R\left[w\Phi(y) - a\right], 0\right\} \ .$$

Then we obtain from (5) and (A3)

$$\Gamma(y,a) = \begin{cases} \mu\left(y,a\right), & \text{if } y \le \underline{y}(a) \\ \mu\left(y,a\right) + \mu_1\left(y,a\right), & \text{if } y \ge \underline{y}(a) \end{cases} \tag{15}$$

By (13) the maximizer

$$y^* := \left(\Phi'\right)^{-1}\left(\frac{1}{Rw}\right) \tag{16}$$

of $\mu\left(y,a\right)$ is constant and coincides with $\widehat{y}_{II}(a)$ for $a \ge \overline{a}$, where

$$\overline{a} := w\Phi\left(y^*\right) \ . \tag{17}$$

For $a \le \overline{a}$, $\widehat{y}_{II}(a)$ is given by

$$y^{**}\left(a\right) := \arg\max_y\left[\mu\left(y,a\right) + \mu_1\left(y,a\right)\right]. \tag{18}$$

To see more precisely what $\mu_1(y,a)$ is, notice that, if (y,a) is such that $2y - R[w\Phi(y) - a] \leq 0$, then there is bankruptcy for sure, and it is clear that in that case, since then $\Gamma(y,a) = 0$, $\mu_1(y,a) = -\mu(y,a)$. If $2y - R[w\Phi(y) - a] \geq 0$, then $\overline{\pi} = 0$ and from (14)

$$\mu_1(y,a) = \frac{1}{4y} R^2 [w\Phi(y) - a]^2 \tag{19}$$

In the Appendix we show that $y^{**}(a)$ is a strictly decreasing function.

To complete the analysis of the decision function $\widehat{y}(a)$ we have to combine $\widehat{y}_{II}(a)$ with the function $\widehat{y}_I(a)$. From $\partial F(y,a)/\partial y > 0$ (see the proof of Lemma 1) it is clear that $\widehat{y}(a)$ is the minimum of $\widehat{y}_I(a)$ and $\widehat{y}_{II}(a)$. Recalling (11) let us now temporarily write α explicitly as an argument of $\widehat{y}_I$, i.e. $\widehat{y}_I(a,\alpha)$. Then observe that by (11) and (12) $\widehat{y}_I(a,0) = \underline{y}(a)$ for all a, $\widehat{y}_I(0,0) = \underline{y}(0) = 0$ and, by (16) to (18), $\widehat{y}_I(\overline{a},0) = \underline{y}(\overline{a}) = y^* = y^{**}(\overline{a})$. Thus, since $\widehat{y}_I$ is increasing in a and y^{**} is decreasing, $\widehat{y}_I(a,0) < y^{**}(a)$ for $a < \overline{a}$. When α is positive, $\widehat{y}_I(a,\alpha) > \underline{y}(a)$ as (11) yields

$$\frac{\partial \widehat{y}_I(a,\alpha)}{\partial \alpha} = \frac{2\widehat{y}_I(a,\alpha)}{Rw\Phi'(\widehat{y}_I(a,\alpha))} > 0 \ .$$

However, by continuity in α, $\widehat{y}_I(0,\alpha) < y^{**}(0)$ for α small enough. Therefore, in that case there exists $\widehat{a} < \overline{a}$ such that $\widehat{y}_I(\widehat{a},\alpha) = y^{**}(\widehat{a})$. Furthermore, by continuity of $\widehat{y}_I(a,\alpha)$ in a there exists $\underline{a} < \widehat{a}$ such that $\widehat{y}_I(\underline{a},\alpha) = y^*$. We summarize these facts in the following proposition.[5]

Proposition 1 *Under assumptions* (A1), (A2) *and* (A3), *for given confidence level* $\alpha \in [0,1]$ *the firm's problem*

$$\begin{gathered} \max\nolimits_y \Gamma(y,a) \\ s.t.\ VaR_\alpha[\Pi(y,a)] \leq 0 \end{gathered}$$

admits a solution $\widehat{y}(a)$ *for any* $a \geq 0$. *Moreover, for* $\alpha > 0$ *but not too large, there exist* $\widehat{a}$ *and* $\overline{a}$ *with* $0 < \widehat{a} < \overline{a}$ *such that* $\widehat{y}(a)$ *is given by*

$$\widehat{y}(a) = \begin{cases} \widehat{y}_I(a) & if \quad a \leq \widehat{a} \\ y^{**}(a) & if \quad \widehat{a} \leq a \leq \overline{a} \\ y^* & if \quad a \geq \overline{a} \end{cases} .$$

The solution is continuous but non-monotone in the equity base a: strictly increasing between zero and $\widehat{a}$, *strictly decreasing between* $\widehat{a}$ *and* $\overline{a}$, *and constant beyond* $\overline{a}$. *This furthermore entails that there exists* $\underline{a} > 0$ *such that the firm's behaviour gives rise to moral hazard whenever* $a \in (\underline{a}, \overline{a})$.[6]

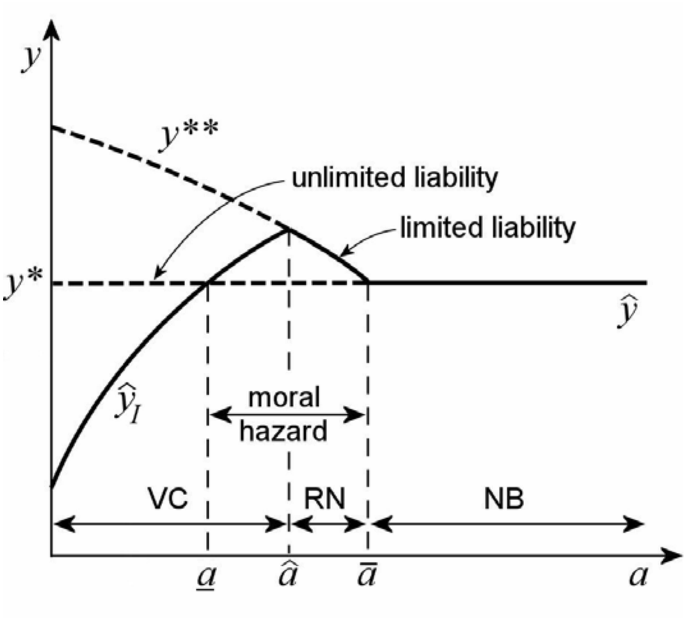

Figure 1: The function $\widehat{y}\left(a\right)$ (thick line)

For an economic explanation of these results let us consider Figure 1 and start with looking at what the firm would do in the case of unlimited liability. Then its objective function would be $\mu\left(y,a\right)$ as defined in (13), and consequently its optimal output y^{*} given in (16) would be constant, *for all* $a\geq 0$. To see why and how things change when limited liability is taken into account, that is the objective function is $\Gamma(y,a)$ as given by (5), let us parametrically diminish a from large values towards zero. For a sufficiently large ($a\geq\overline{a}$) nothing changes relative to the case with unlimited liability because for $a\geq\overline{a}$ there is no risk of bankruptcy (region NB).

When a is below $\overline{a}$, more precisely in region RN (risk neutrality), the probability of bankruptcy while producing $y=y^{*}$ is positive and, if bankruptcy occured, with unlimited liability the firm would suffer a loss. With

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[5] This is a simplified version of the analogous result in Tulli and Weinrich (2009).

[6] We refer to the common definition of *moral hazard* according to which it is the prospect that a party insulated from risk may behave differently from the way it would behave if it were fully exposed to the risk. Moral hazard arises because an individual or institution does not bear the full consequences of its actions, and therefore has a tendency to act less carefully than it otherwise would, leaving another party to bear responsibility for the consequences of those actions.

limited liability, the loss is smaller (with zero bankruptcy cost it is zero), and the firm has an incentive to run a higher risk of bankruptcy, which is tantamount to producing a quantity larger than y^*. Therefore $y^{**}(a) > y^*$ for $a < \overline{a}$. With increasing output - i.e. decreasing a - the probability of bankruptcy increases until at $\widehat{a}$ it reaches the confidence level α. At this point the VaR-constraint becomes binding (region VC), and with equity a further diminished, output $\widehat{y}_I(a)$ decreases, too. But as long as $a > \underline{a}$, $\widehat{y}_I(a) > y^*$, and the probability of bankruptcy is larger than while producing y^*. Thus for $\underline{a} < a < \overline{a}$ there is moral hazard, whereas for $a \leq \underline{a}$ this is no longer true.

Of course, moral hazard could be avoided by setting the confidence level α to zero in which case $\underline{a} = \widehat{a} = \overline{a}$, but that might severely hamper the firm's business prospects. In particular, if the firm had zero equity, since $\widehat{y}_I(0) = 0$ in case $\alpha = 0$, the firm, from a dynamic point of view, could not take off while with $\alpha > 0$ it can. A more satisfactory solution to the the moral hazard problem will be achieved in the dynamic framework of the next section where it will be shown that, also if α is positive, the firm chooses, induced by the VaR-constraint, the equity base a so that no moral hazard occurs.

Example 1 Assume $\Phi(y) = ky^2$, $k > 0$. Then it can easily be checked that

$$\widehat{y}_I(a) = \frac{\alpha + \sqrt{\alpha^2 + R^2wka}}{Rwk}, \; y^* = \frac{1}{2Rwk}, \; \underline{a} = \frac{1-4\alpha}{4R^2wk}, \; \overline{a} = \frac{1}{4R^2wk}.$$

Moreover, it can be shown that $y^{**}(a)$ is that real solution to the equation

$$3(Rwk)^2 y^4 - 8Rwky^3 + \left(4 - 2R^2wka\right) y^2 - (Ra)^2 = 0$$

which gives the highest value to the function $\Gamma(\cdot, a)$; it is[7]

$$y^{**}(a) = \frac{1 + \sqrt{1 - 3R^2wka}}{3Rwk}, \tag{20}$$

and then

$$\widehat{a} = \frac{1 - 2\alpha - 3\alpha^2}{4R^2wk}.$$

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[7] See Tulli (2004), eq. (2.20), p. 67.

3 Dynamic Analysis

3.1 The problem

We now extend the previous analysis by embedding it in a dynamic framework. The realized profit can be seen as the outcome of a particular period t, $\pi = \pi_t$, giving rise to the next period's equity base $a_{t+1} = (1-d_t)\,\pi_t$, where $d_t \in [0,1]$ is the share of profits that is paid out to shareholders as dividends, and $d_t\pi_t = \pi_t - a_{t+1}$ is its total value. This raises the question of the optimal choice of retained capital and the accumulated value of dividends paid over time, and its dependence on the interest rate, the wage rate, the confidence level and the time-preference or discount rate of shareholders.

Let $v(a) := \Gamma(\widehat{y}\,(a)\,,a)$ denote the value function of the static problem. Then by Proposition 1

$$v(a) = \begin{cases} \Gamma(\widehat{y}_I(a), a) & \text{if } a \leq \widehat{a} \\ \\ \Gamma(y^{**}\,(a)\,, a) & \text{if } \widehat{a} \leq a \leq \overline{a} \\ \\ \Gamma(y^{*}, a) & \text{if } a \geq \overline{a} \end{cases} \tag{21}$$

We assume that the operation of the firm in any period t is guided by the managers' objective to maximize $\Gamma(\cdot, a_t)$. Then the expected dividend payment to shareholders in that period is $v\,(a_t) - a_{t+1}$, provided the firm has survived until then. The shareholders have to choose for any t a dividend share $d_t = [v\,(a_t) - a_{t+1}]\,/v\,(a_t)$ or, equivalently, an amount of equity a_t. This choice results from solving the problem

$$\begin{gathered} \max_{(a_t)_{t\geq 0}} \; \textstyle\sum_{t=0}^{\infty} \beta'_t \left[v\,(a_t) - a_{t+1}\right] \\ s.t.\; a_{t+1} \in [0, v\,(a_t)] \;\; \forall t \geq 0,\;\; a_0 \text{ given,} \end{gathered}$$

where $\beta'_t = \Pi_{\tau=0}^{t}\beta_\tau$, and β_τ, $\tau = 0, 1, \ldots$, are (one-period) discount factors to be determined later. In particular, they will have to capture the possibility of bankruptcy. Of course, if $\beta_\tau = \overline{\beta}$ for all τ, then $\beta'_t = \overline{\beta}^{t}$.

The necessary Euler-Lagrange condition for a solution $a^* = (a^*_t)_{t\geq 1}$ is, setting $u\,(a_t, a_{t+1}) := v\,(a_t) - a_{t+1}$,

$$u\left(a^*_{t-1}, a^*_t\right) + \beta_t u\left(a^*_t, a^*_{t+1}\right) \geq u\left(a^*_{t-1}, a_t\right) + \beta_t u\left(a_t, a^*_{t+1}\right) \tag{22}$$

for any $a_t \in \left[0, v\left(a^*_{t-1}\right)\right]$ with $a^*_{t+1} \in [0, v\,(a_t)]$ $\forall t \geq 1$. Then an interior solution satisfies

$$u_2\left(a^*_{t-1}, a^*_t\right) + \beta_t u_1\left(a^*_t, a^*_{t+1}\right) = 0 \;\forall t \geq 1. \tag{23}$$

In the present set-up this yields

$$-1+\beta_t v'\left(a_t^*\right)=0\ \forall t\geq 1. \tag{24}$$

Thus, in case β_t is constant, $\beta_t=\overline{\beta}$, so is a_t^*, i.e. $a_t^*=\overline{a}^*$ for all $t\geq 1$.

3.2 The discount factors

Let us now determine the discount factors β_t' and β_τ. Since in any period τ there is a possibility of going bankrupt, say $\delta_\tau\in[0,1]$, the probability of survival until period $t\geq 2$ is $\Pi_{\tau=1}^{t-1}\left(1-\delta_\tau\right)$. Thus the dividend payment in period t, $v\left(a_t\right)-a_{t+1}$, has to be multiplied by this probability to obtain its expected value. Moreover, the shareholders of the firm may have a constant subjective one-period discount factor $\beta\leq 1$. Then the effective discount factors to be applied to the outcome of any period t, $v\left(a_t\right)-a_{t+1}$, as seen from the initial period, are

$$\beta_0'=1,\ \beta_1'=\beta,\ \beta_t'=\beta^t\Pi_{\tau=1}^{t-1}\left(1-\delta_\tau\right)\ \ \forall t\geq 2.$$

More precisely still, the probability of bankruptcy in any given period $t\geq 1$ depends on a_{t-1} and is

$$\delta_t=\delta\left(a_{t-1}\right):=\begin{cases}\alpha & \text{if } a_{t-1}\leq\widehat{a}\\ F\left(y^{**}(a_{t-1}),a_{t-1}\right) & \text{if } \widehat{a}\leq a_{t-1}\leq\overline{a}\\ 0 & \text{if } a_{t-1}\geq\overline{a}\end{cases}$$

Since $0\leq F\left(y^{**}(a),a\right)\leq\alpha$, $1-\delta\left(a\right)\geq 1-\alpha$ for any $a\in[\widehat{a},\overline{a}]$. With these specifications, the coefficients β_t in (22)-(24) now become

$$\beta_0=1,\ \beta_1=\beta \text{ and } \beta_t=\beta_t'/\beta_{t-1}'=\beta\left(1-\delta\left(a_{t-2}\right)\right)\ \forall t\geq 2.$$

3.3 Determination of $v'(a)$

To explore condition (24), we next have to determine $v'(a)$. From (6) and by the envelope theorem,

$$v'(a)=\frac{\partial\Gamma(\widehat{y}(a),a)}{\partial a}-\lambda\left(a\right)\frac{\partial F\left(\widehat{y}(a),a\right)}{\partial a}$$

where $\lambda\left(a\right)\geq 0$ is the value of the Lagrange-multiplier of the VaR-constraint at the solution of the static problem. More precisely, we have, from (21) and using (15), (13) and (19),

$$v'(a) = \begin{cases} \frac{\partial \mu}{\partial a}\left(\widehat{y}_I(a), a\right) + \frac{\partial \mu_1}{\partial a}\left(\widehat{y}_I(a), a\right) - \lambda\left(a\right)\frac{\partial F}{\partial a}\left(\widehat{y}_I(a), a\right) & \text{if} \quad a < \widehat{a} \\ \frac{\partial \mu}{\partial a}\left(y^{**}, a\right) + \frac{\partial \mu_1}{\partial a}\left(y^{**}(a), a\right) & \text{if} \quad \widehat{a} < a < \overline{a} \\ \frac{\partial \mu}{\partial a}\left(y^{*}, a\right) & \text{if} \quad a > \overline{a} \end{cases} .$$

$$= \begin{cases} R - \frac{R^2}{2\widehat{y}_I(a)}[w\Phi(\widehat{y}_I(a)) - a] + \lambda(a)\frac{R}{2\widehat{y}_I(a)} & \text{if} \quad a < \widehat{a} \\ R - \frac{R^2}{2y^{**}(a)}[w\Phi(y^{**}(a)) - a] & \text{if} \quad \widehat{a} < a < \overline{a} \\ R & \text{if} \quad a > \overline{a} \end{cases} \qquad (25)$$

Since $\widehat{y}_I(\widehat{a}) = y^{**}(\widehat{a})$ and $\lambda(\widehat{a}) = 0$, $v'^{-}(\widehat{a}) = v'^{+}(\widehat{a})$. Also, $y^{**}(\overline{a}) = y^{*}$ and $w\Phi(y^{*}) - \overline{a} = 0$ imply $v'^{-}(\overline{a}) = v'^{+}(\overline{a})$. Thus $v(a)$ is differentiable. Moreover, since $w\Phi(y^{**}(a)) - a > 0$ for $a < \overline{a}$, $v'(a_1) < R = v'(a_2)$ for all $a_1 \in [\widehat{a}, \overline{a}]$, $a_2 > \overline{a}$.

Consider now what happens to $\lambda(a)$ as $a \to 0$. Since the VaR-constraint is binding for $a < \widehat{a}$, $\lambda(a) > 0$ in that case. Moreover, from the Kuhn-Tucker conditions,

$$\frac{\partial \Gamma(\widehat{y}_I(a), a)}{\partial y} - \lambda\left(a\right)\frac{\partial F\left(\widehat{y}_I(a), a\right)}{\partial y} = 0$$

and thus

$$\lambda\left(a\right) = \frac{\frac{\partial \Gamma(\widehat{y}_I(a), a)}{\partial y}}{\frac{\partial F(\widehat{y}_I(a), a)}{\partial y}} .$$

From (13) and (19) we get

$$\frac{\partial \Gamma(y,a)}{\partial y} = 1 - Rw\Phi'\left(y\right) - \frac{R^2}{4y^2}\left[w\Phi(y) - a\right]^2 + \frac{R^2}{2y}\left[w\Phi(y) - a\right]w\Phi'(y) .$$

If $y = \widehat{y}_I(a)$, then by (11)

$$\begin{aligned} \frac{\partial \Gamma(y,a)}{\partial y} &= 1 - Rw\Phi'\left(y\right) - \frac{\left(2\alpha y\right)^2}{4y^2} + \frac{R}{2y}2\alpha y w\Phi'(y) \\ &= 1 - \alpha^2 - \left(1 - \alpha\right)Rw\Phi'\left(y\right) \end{aligned}$$

On the other hand, from (10)

$$F(y,a) = \frac{R}{2y}\left[w\Phi(y) - a\right] \qquad (26)$$

which implies

$$\frac{\partial F}{\partial a} = -\frac{R}{2y} \tag{27}$$

and, for $y = \widehat{y}_I(a)$,

$$\begin{aligned}
\frac{\partial F}{\partial y}(y,a) &= \frac{R}{2}\frac{yw\Phi'(y) - w\Phi(y) + a}{y^2} = -\frac{R}{2y}\left[w\Phi(y) - a\right]\frac{1}{y} + \frac{Rw\Phi'(y)}{2y} \\
&= -\frac{2\alpha y}{2y}\frac{1}{y} + \frac{Rw\Phi'(y)}{2y} = -\frac{\alpha}{y} + \frac{Rw\Phi'(y)}{2y} \\
&= \frac{Rw\Phi'(y) - 2\alpha}{2y} \ .
\end{aligned}$$

Therefore

$$\lambda(a) = \frac{1-\alpha^2 - (1-\alpha)\,Rw\Phi'(y)}{Rw\Phi'(y) - 2\alpha}\cdot 2y \ .$$

Inserting in (25) and using (27) yields

$$\begin{aligned}
v'(a) &= R - \frac{R^2}{2\widehat{y}_I(a)}\left[w\Phi(\widehat{y}_I(a)) - a\right] + \frac{1-\alpha^2 - (1-\alpha)\,Rw\Phi'(\widehat{y}_I(a))}{Rw\Phi'(\widehat{y}_I(a)) - 2\alpha}R \\
&= R\left[1 - \alpha + \frac{1-\alpha^2 - (1-\alpha)\,Rw\Phi'(\widehat{y}_I(a))}{Rw\Phi'(\widehat{y}_I(a)) - 2\alpha}\right] \\
&= \frac{R}{Rw\Phi'(\widehat{y}_I(a)) - 2\alpha}\left[-(1-\alpha)\,2\alpha + 1 - \alpha^2\right] \\
&= (1-\alpha)^2\,\frac{R}{Rw\Phi'(\widehat{y}_I(a)) - 2\alpha}
\end{aligned}$$

Using that (11) implies

$$\widehat{y}_I'(a) = \frac{R}{Rw\Phi'(\widehat{y}_I(a)) - 2\alpha} \tag{28}$$

we get

$$v'(a) = (1-\alpha)^2\,\widehat{y}_I'(a) \ . \tag{29}$$

By (11) and (28) we can rewrite $\widehat{y}_I'(a)$ as

$$\begin{aligned}
\widehat{y}_I'(a) &= \frac{R}{Rw\Phi'(\widehat{y}_I(a)) - \frac{R[w\Phi(\widehat{y}_I(a)) - a]}{\widehat{y}_I(a)}} \\
&= \frac{\widehat{y}_I(a)}{w\Phi'(\widehat{y}_I(a))\widehat{y}_I(a) - [w\Phi(\widehat{y}_I(a)) - a]} \ .
\end{aligned}$$

In particular

$$\widehat{y}_I'(0) = \frac{1}{w\Phi'(\widehat{y}_I(0)) - \frac{w\Phi(\widehat{y}_I(0))}{\widehat{y}_I(0)}} \ . \tag{30}$$

Moreoevor, (11) yields

$$Rw\frac{\Phi(\widehat{y}_I(0))}{\widehat{y}_I(0)} = 2\alpha \ . \tag{31}$$

Therefore, if $\alpha \to 0$, then $\Phi(\widehat{y}_I(0))/\widehat{y}_I(0) \to 0$ which implies $\Phi(\widehat{y}_I(0)) \to 0$, and thus $\widehat{y}_I(0) \to 0$ by (A2). Again by (A2), this means $\Phi'(\widehat{y}_I(0)) \to 0$, and therefore $\widehat{y}_I'(0) \to \infty$. Thus $\lim_{\alpha\to 0} v'(0) = \infty$ and, for small $\alpha > 0$, $v'(0)$ is large.

For example, with $\Phi(y) = ky^2$

$$\widehat{y}_I'(0) = \frac{1}{w\left[2k\widehat{y}_I(0) - \frac{k(\widehat{y}_I(0))^2}{\widehat{y}_I(0)}\right]} = \frac{1}{wk\widehat{y}_I(0)}$$

and

$$Rwk\widehat{y}_I(0) = 2\alpha \Leftrightarrow \widehat{y}_I(0) = \frac{2\alpha}{Rwk}$$

which implies

$$\widehat{y}_I'(0) = \frac{1}{wk\frac{2\alpha}{Rwk}} = \frac{R}{2\alpha} \ . \tag{32}$$

Then

$$v'(0) = (1-\alpha)^2\,\widehat{y}_I'(0) = \frac{(1-\alpha)^2}{2\alpha}R \ .$$

Using for example $\alpha = 0.01$ and $R = 1.1$, this yields $v'(0) = 53.9055$.

3.4 Solution of the dynamic problem

Let us now come back to the dynamic optimality condition (24). We distinguish between:

- $a_t < \widehat{a}$, the VC region (VaR-constrained),
- $\widehat{a} < a_t < \overline{a}$, the RN region (risk neutrality), and
- $a_t > \overline{a}$, the NB region (no bankruptcy).

Consider first the case that $a_t \in NB$. Then $\delta(a_t) = 0$, $\beta_t = \beta$ and hence, if there were an optimal policy (a_t), it should be constant. Since $v'(a_t) = R$, (24) becomes

$$-1 + \beta R = 0 \ .$$

Unless in the unlikely case that $\beta = 1/R$, this condition can obviously not be fulfilled. If $\beta R > 1$, then a_t should be increased without limit. If, on the contrary, $\beta R < 1$, then

$$u_2\left(a_{t-1}, a_t\right) + \beta u_1\left(a_t, a_{t+1}\right) = -1 + \beta R < 0$$

for all t, and hence it is always convenient to decrease a_t. This leads the firm to exit the NB region. Thus we have the following result:

Proposition 2 *In case the subjective discount factor β is smaller than $1/R$, it is not dynamically optimal to stay in the region where there is no risk of bankruptcy.*

Next consider the case that $a_t \in RN$. Then from (25), since $w\Phi(y^{**}(a)) - a > 0$, $v'(a) < R$. Moreover,

$$\beta_t = \beta\left(1 - \delta\left(a_{t-2}\right)\right) < \beta \ ,$$

and hence

$$-1 + \beta_t v'(a) < 0 \text{ if } \beta R > 1 \ .$$

Thus also in this region no optimal a_t exists; it would be reduced until the region RN has been left.

Finally consider the region VC. There $v'(a)$ varies between $v'(\widehat{a})$ and $v'(0)$. Since $\lambda(\widehat{a}) = 0$,

$$v'(\widehat{a}) = R - \frac{R^2}{2\widehat{y}_I(a)}[w\Phi(\widehat{y}_I(\widehat{a})) - \widehat{a}] < R$$

as $w\Phi(\widehat{y}_I(\widehat{a})) - \widehat{a} > 0$. Moreover,

$$\beta_t = \beta\left(1 - \alpha\right) < \beta \ .$$

Thus

$$-1 + \beta_t v'(\widehat{a}) < 0 \text{ if } \beta\left(1 - \alpha\right) R < 1,$$

and it is convenient to diminish a_t below $\widehat{a}$. On the other hand, as shown above, $v'(0)$ is large, and thus, as illustrated in Figure 2, there may exist $\overline{a}^* \in VC$ such that

$$-1 + \beta\left(1 - \alpha\right) v'\left(\overline{a}^*\right) = 0 \ .$$

From this follows that a sufficient condition for $\overline{a}^* > 0$ is

$$-1 + \beta\left(1 - \alpha\right) v'(0) > 0 \tag{33}$$

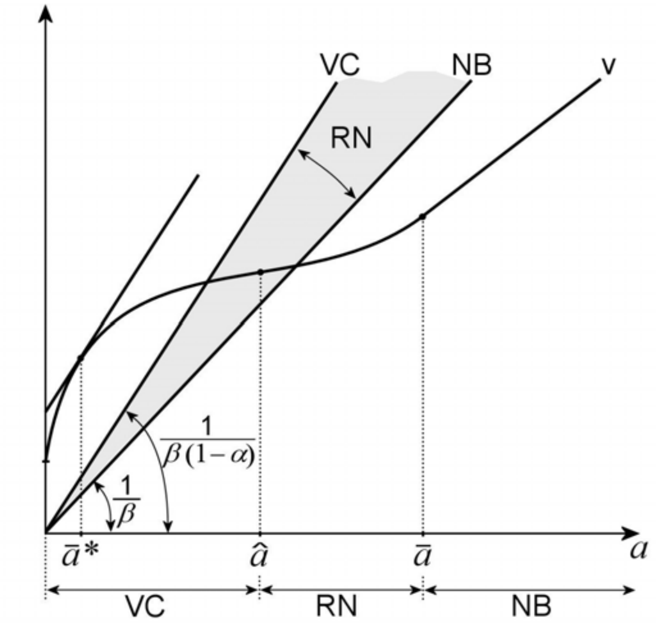

Figure 2: Graphical characterization of $\overline{a}^*$.

$\Leftrightarrow$ (by (29))

$$-1+\beta\,(1-\alpha)^3\,\widehat{y}_I'(0)>0$$

$\Leftrightarrow$ (by (30))

$$-1+\frac{\beta\,(1-\alpha)^3\,\widehat{y}_I(0)}{w\Phi'(\widehat{y}_I(0))\widehat{y}_I(0)-w\Phi(\widehat{y}_I(0))}>0$$

$\Leftrightarrow$

$$\beta\,(1-\alpha)^3>w\left[\Phi'(\widehat{y}_I(0))-\frac{\Phi(\widehat{y}_I(0))}{\widehat{y}_I(0)}\right]$$

$\Leftrightarrow$

$$\Phi'(\widehat{y}_I(0))-\frac{\Phi(\widehat{y}_I(0))}{\widehat{y}_I(0)}<\frac{\beta\,(1-\alpha)^3}{w}$$

$\Leftrightarrow$

$$\Phi'(\widehat{y}_I(0))\frac{\widehat{y}_I(0)}{\Phi(\widehat{y}_I(0))}-1<\frac{\beta\,(1-\alpha)^3}{w}\frac{\widehat{y}_I(0)}{\Phi(\widehat{y}_I(0))}\ . \tag{34}$$

Note that we can express this inequality in terms of the *elasticity of scale* which, denoting the production function $y=f\,(n):=\Phi^{-1}\,(n)$, is

$$f'\,(n)\,\frac{n}{f\,(n)}=\frac{1}{\Phi'(y)}\frac{\Phi(y)}{y}=:\varepsilon_s\,(y)\ .$$

Thus (34) is equivalent to

$$\frac{1}{\varepsilon_s\left(\widehat{y}_I(0)\right)} < 1 + \frac{\beta\left(1-\alpha\right)^3}{w}\frac{\widehat{y}_I(0)}{\Phi(\widehat{y}_I(0))}$$

which, using (31), yields

$$\varepsilon_s\left(\widehat{y}_I(0)\right) > \left[1 + \frac{\left(1-\alpha\right)^3}{2\alpha}\beta R\right]^{-1} . \tag{35}$$

Since Φ is strictly convex with $\Phi\left(0\right) = 0$, $\Phi'(y) > \Phi(y)/y$ for all y, which implies $\varepsilon_s\left(y\right) < 1$ for all y.

For example, if $\Phi(y) = ky^2$, then $\varepsilon_s\left(y\right) = 1/2$ for all y; using $\alpha = 0.01$, $\beta = 0.9$ and $R = 1.1$, we get $\left[1 + \frac{(1-\alpha)^3}{2\alpha}\beta R\right]^{-1} = 0.0204$, which is way below the value for ε_s. This suggests that the sufficient condition for $\overline{a}^* > 0$ is normally satisfied. In terms of a labour requirement function $\Phi(y) = ky^\gamma$, $k, \gamma > 0$, in which case $\varepsilon_s\left(y\right) = 1/\gamma$ for all y, this means $\gamma < 1/0.0204 = 49.0196$.

Note that condition (35) can be verified without knowing the value of $\widehat{y}_I(0)$ if sufficient information is available about the elasticity function $\varepsilon_s\left(\cdot\right)$, for instance that it is constant. As just mentioned, that is the case if the labour requirement function is of the form $\Phi(y) = ky^\gamma$. Note also that (35) can never be fulfilled if $\beta = 0$, that is if shareholders do not care at all about any period other than the present one. In fact, it is clear that in that case $\overline{a}^* = 0$ since the shareholders choose $d_t = 1$ in any period t (as long as the firm exists). We summarize the results in the following

Proposition 3 *For a subjective discount factor β such that $\beta R < 1$, the shareholders of the firm will adopt a constant policy of retained equity $\overline{a}^* \geq 0$ lying in the region VC (i.e. the VaR-constraint is binding) and satisfying $v'\left(\overline{a}^*\right) \leq \left[\beta\left(1-\alpha\right)\right]^{-1}$, with*

$$v'\left(\overline{a}^*\right) = \frac{1}{\beta\left(1-\alpha\right)} \text{ if } \overline{a}^* > 0 . \tag{36}$$

A sufficient condition for $\overline{a}^*$ to be positive is that the firm's elasticity of scale, $\varepsilon_s\left(y\right) = \left(1/\Phi'(y)\right)\left(\Phi(y)/y\right)$, is not too small at $\widehat{y}_I(0)$, i.e.

$$\varepsilon_s\left(\widehat{y}_I(0)\right) > \left[1 + \frac{\left(1-\alpha\right)^3}{2\alpha}\beta R\right]^{-1} . \tag{37}$$

Otherwise $\overline{a}^*$ may be zero, for example if β is close to zero. That would also be the outcome if the firm were not subjected to the VaR-constraint.

3.5 Comparative statics

We now want to determine the effect of parameter variations on the dynamically optimal values $\overline{a}^*$ and $\overline{y}^* := \widehat{y}_I(\overline{a}^*, R, w, \alpha)$, where $\widehat{y}_I(a, R, w, \alpha)$ is defined from (11) implicitly by

$$\Phi\left(\widehat{y}_I(a, R, w, \alpha)\right) = \frac{2\alpha}{Rw}\widehat{y}_I(a, R, w, \alpha) + \frac{a}{w} \ . \tag{38}$$

Note that strict positive monotonicity and strict convexity of Φ, and $\Phi(0) = 0$, imply

$$\Phi'\left(\widehat{y}_I(a, R, w, \alpha)\right) > \frac{2\alpha}{Rw} \ . \tag{39}$$

By (29) and (36) we have

$$\frac{\partial \widehat{y}_I}{\partial a}(\overline{a}^*, R, w, \alpha) = \frac{1}{\beta\left(1-\alpha\right)^3} \text{ if } \overline{a}^* > 0 \ . \tag{40}$$

Since from (38)

$$\frac{\partial \widehat{y}_I}{\partial a}(a, R, w, \alpha) = \frac{1}{w\Phi'\left(\widehat{y}_I(a, \alpha, R, w)\right) - 2\alpha/R} \tag{41}$$

the optimality condition (40) yields for $\overline{y}^*$

$$w\Phi'\left(\overline{y}^*\right) - \frac{2\alpha}{R} = \beta\left(1-\alpha\right)^3 \ , \tag{42}$$

which is thus a function $\overline{y}^*\left(\beta, R, w, \alpha\right)$. It yields the following comparative-statics results.

Proposition 4 *The dynamically optimal level of output reacts to parameter variations as follows:*

$$\frac{\partial \overline{y}^*}{\partial \beta} > 0, \frac{\partial \overline{y}^*}{\partial R} < 0, \frac{\partial \overline{y}^*}{\partial w} < 0 \ .$$

Moreover,

$$\frac{\partial \overline{y}^*}{\partial \alpha} \begin{cases} < 0 \textit{ for } \beta > \underline{\beta}\left(\alpha\right) \\ > 0 \textit{ for } \beta < \underline{\beta}\left(\alpha\right) \end{cases} \tag{43}$$

where

$$\underline{\beta}\left(\alpha\right) := \frac{2}{3R\left(1-\alpha\right)^2} \tag{44}$$

and

$$\underline{\beta}\left(\alpha\right) < 1 \textit{ for } \alpha < 1 - \sqrt{2/3} \approx 0.1835 \ . \tag{45}$$

Proof. Differentiating (42) implicitly yields

$$\frac{\partial \overline{y}^*}{\partial \beta} = \frac{(1-\alpha)^3}{w\Phi''(\overline{y}^*)}$$

which is positive by strict convexity of Φ. Similarly,

$$\frac{\partial \overline{y}^*}{\partial R} = -\frac{2\alpha/R^2}{w\Phi''(\overline{y}^*)} < 0$$

and

$$\frac{\partial \overline{y}^*}{\partial w} = -\frac{\Phi'(\overline{y}^*)}{w\Phi''(\overline{y}^*)} < 0 \ .$$

Finally,

$$\frac{\partial \overline{y}^*}{\partial \alpha} = -\frac{-2/R + 3\beta(1-\alpha)^2}{w\Phi''(\overline{y}^*)} = \frac{2/R - 3\beta(1-\alpha)^2}{w\Phi''(\overline{y}^*)}$$

which is negative iff

$$\frac{2}{R} - 3\beta(1-\alpha)^2 < 0$$

$\Leftrightarrow$

$$\beta > \frac{2}{3R(1-\alpha)^2} =: \underline{\beta}\ (\alpha)\,.$$

Since $R > 1$, $\alpha < 1 - \sqrt{2/3}$ implies $\underline{\beta}\ (\alpha) < 1$. ■

Comparing with the results of the static analysis ($\partial\widehat{y}_I/\partial R < 0, \partial\widehat{y}_I/\partial w < 0, \partial\widehat{y}_I/\partial\alpha > 0$), we see that the short-run effects of variations in the interest rate and of the wage rate are confirmed in a dynamic, long-run perspective. This is not true for the confidence level α: while in the short-run (which can be identified with the case $\beta = 0$) an increase in α prompts the firm to increase output as the VaR-constraint is relaxed, in the long-run it has a negative effect as long as the subjective discount rate β is sufficiently high, i.e. the shareholders sufficiently farsighted. (For example with $R = 1.1$ $\underline{\beta}(0.01) = 0.618366$.) This is because an increase in α increases the probability of bankruptcy and thus decreases the effective discount rate $\beta(1-\alpha)$ which, according to $\partial\overline{y}^*/\partial\beta > 0$, reduces the optimal level of output - but only if β is large enough to give sufficient weight to the decrease in $1-\alpha$.

There remain to be determined the effects of parameter variations on the optimal equity level $\overline{a}^*$. This turns out to be more complex than the corresponding behaviour of $\overline{y}^*$. The results of the previous proposition give some indication what is involved. Indeed, from (38) we have

$$\overline{a}^* = w\Phi\left(\overline{y}^*\right) - \frac{2\alpha}{R}\overline{y}^* \ . \tag{46}$$

Thus

$$\frac{\partial \overline{a}^*}{\partial \beta} = \left[w\Phi'\left(\overline{y}^*\right) - \frac{2\alpha}{R}\right]\frac{\partial \overline{y}^*}{\partial \beta} > 0 \tag{47}$$

while

$$\frac{\partial \overline{a}^*}{\partial R} = \left[w\Phi'\left(\overline{y}^*\right) - \frac{2\alpha}{R}\right]\frac{\partial \overline{y}^*}{\partial R} + \frac{2\alpha}{R^2}\overline{y}^* \ , \tag{48}$$

$$\frac{\partial \overline{a}^*}{\partial w} = \Phi\left(\overline{y}^*\right) + \left[w\Phi'\left(\overline{y}^*\right) - \frac{2\alpha}{R}\right]\frac{\partial \overline{y}^*}{\partial w} \tag{49}$$

and

$$\frac{\partial \overline{a}^*}{\partial \alpha} = \left[w\Phi'\left(\overline{y}^*\right) - \frac{2\alpha}{R}\right]\frac{\partial \overline{y}^*}{\partial \alpha} - \frac{2}{R}\overline{y}^* \tag{50}$$

are all expressions which which contain both positive and negative terms because there are contrasting direct and indirect effects. For example, in (49) the first term on the right hand side is positive and represents the marginal increase in labour cost due to an increase in the wage rate; the second term is negative and expresses the decrease in output due to the increase in production cost. Therefore a more sophisticated analysis is called for the results of which are summarized in the following proposition.

Proposition 5 *The dynamically optimal value of the equity base $\overline{a}^*$ reacts to parameter variations as follows:*

(a) Discount factor:

$$\frac{\partial \overline{a}^*}{\partial \beta} > 0 \ .$$

(b) Interest rate: the effect of a change in R is ambiguous; however, if the elasticity of scale ε_s is constant, then there is a critical value $\overline{\varepsilon}_s \in [1/2, 1)$ such that

$$\frac{\partial \overline{a}^*}{\partial R}\begin{cases} < 0 \\ > 0 \end{cases} \Leftrightarrow \ \varepsilon_s \begin{cases} > \overline{\varepsilon}_s \\ < \overline{\varepsilon}_s \end{cases} .$$

Moreover, $\overline{\varepsilon}_s > 1/2$ when $\alpha > 0$ and $lim_{\alpha \to 0}\overline{\varepsilon}_s = 1/2$.

(c) Wage rate: if the elasticity of scale ε_s is constant, then

$$\frac{\partial \overline{a}^*}{\partial w} < 0 \ .$$

(d) Confidence level: a sufficient condition for

$$\frac{\partial \overline{a}^*}{\partial \alpha} < 0$$

is $\beta > \underline{\beta}(\alpha)$ *where* $\underline{\beta}(\alpha)$ *is as defined in (44).*

Proof. a) Follows from (47) due to (39) and Proposition 4.

b) $\overline{y}^* = \widehat{y}_I(\overline{a}^*, R, w, \alpha)$ and (42) yield

$$w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - \frac{2\alpha}{R} = \beta\left(1-\alpha\right)^3 \ . \tag{51}$$

Therefore

$$\frac{\partial \overline{a}^*}{\partial R} = -\frac{w\Phi''\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\left(\partial \widehat{y}_I/\partial R\right) + 2\alpha/R^2}{w\Phi''\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\left(\partial \widehat{y}_I/\partial a\right)}$$

and thus

$$sgn\left(\frac{\partial \overline{a}^*}{\partial R}\right) = -sgn\left(w\Phi''\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\left(\partial \widehat{y}_I/\partial R\right) + 2\alpha/R^2\right) \ . \tag{52}$$

From (38)

$$\frac{\partial \widehat{y}_I}{\partial R} = -\frac{\left(2\alpha/R^2\right)\widehat{y}_I(\overline{a}^*, R, w, \alpha)}{w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)} \ .$$

If $\varepsilon_s =$ constant, then

$$\Phi\left(y\right) = ky^{\gamma}, \Phi'\left(y\right) = \gamma k y^{\gamma-1}, \Phi''\left(y\right) = \gamma\left(\gamma-1\right) k y^{\gamma-2} \ .$$

Therefore

$$\begin{aligned}
& w\Phi''\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\left(\partial \widehat{y}_I/\partial R\right) + 2\alpha/R^2 \\
= \ & w\gamma\left(\gamma-1\right) k\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)^{\gamma-2} \frac{-\left(2\alpha/R^2\right)\widehat{y}_I(\overline{a}^*, R, w, \alpha)}{w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)} + 2\alpha/R^2 \\
= \ & w\left(\gamma-1\right)\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) \frac{-2\alpha/R^2}{w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)} + 2\alpha/R^2 \\
= \ & \frac{\left(2\alpha/R^2\right) w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\left(1-(\gamma-1)\right) - \left(2\alpha/R^2\right)(2\alpha/R)}{w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)} \\
= \ & \left(2\alpha/R^2\right) \frac{(2-\gamma)\, w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)}{w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)} \ .
\end{aligned}$$

We know from (39) that $w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R) > 0$. Thus there is a unique $\overline{\gamma} \in (1, 2]$ such that

$$(2-\overline{\gamma})\, w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R) = 0\ .$$

Moreover, $\overline{\gamma} < 2$ if $\alpha > 0$ and $\lim_{\alpha\to 0}\overline{\gamma} = 2$. Consequently, by (52) and since the elasticity of scale is $\varepsilon_s = 1/\gamma$, we obtain

$$\frac{\partial \overline{a}^*}{\partial R}\left\{\begin{array}{c} >0 \\ <0 \end{array}\right. \Leftrightarrow 2-\gamma\left\{\begin{array}{c} <2-\overline{\gamma} \\ >2-\overline{\gamma} \end{array}\right. \Leftrightarrow \gamma\left\{\begin{array}{c} >\overline{\gamma} \\ <\overline{\gamma} \end{array}\right. \Leftrightarrow \varepsilon_s\left\{\begin{array}{c} <1/\overline{\gamma} \\ >1/\overline{\gamma} \end{array}\right. .$$

c) From (51) we get

$$\frac{\partial \overline{a}^*}{\partial w} = -\frac{\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) + w\Phi''\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\left(\partial\widehat{y}_I/\partial w\right)}{w\Phi''\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\left(\partial\widehat{y}_I/\partial a\right)}$$

and thus

$$sgn\left(\frac{\partial \overline{a}^*}{\partial w}\right) = -sgn\left(\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) + w\Phi''\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\left(\partial\widehat{y}_I/\partial w\right)\right)\ . \tag{53}$$

From (38)

$$\frac{\partial \widehat{y}_I}{\partial w} = \frac{-\Phi\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)}{w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)}.$$

Therefore, if $\varepsilon_s =$ constant, then

$$\begin{array}{cl}
& \Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) + w\Phi''\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\left(\partial\widehat{y}_I/\partial w\right) \\
= & \left[w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)\right]^{-1} \\
& \times\left[\begin{array}{c} w\left(\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\right)^2 \\ -\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)(2\alpha/R) \\ -w\Phi''\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)\Phi\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) \end{array}\right] \\
= & \left[w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)\right]^{-1} \\
& \times\left[\begin{array}{c} w\left(\gamma k\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)^{\gamma-1}\right)^2 \\ -\gamma k\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)^{\gamma-1}(2\alpha/R) \\ -w\gamma(\gamma-1)\,k\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)^{\gamma-2} k\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)^{\gamma} \end{array}\right] \\
= & \left[w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)\right]^{-1} \\
& \times\left[\begin{array}{c} wk^2\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)^{2\gamma-2}\left(\gamma^2-\gamma^2+\gamma\right) \\ -\gamma k\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)^{\gamma-1}(2\alpha/R) \end{array}\right] \\
= & \left[w\Phi'\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right) - (2\alpha/R)\right]^{-1} \\
& \times\gamma k\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)^{\gamma-1}\left[wk\left(\widehat{y}_I(\overline{a}^*, R, w, \alpha)\right)^{\gamma-1} - (2\alpha/R)\right]
\end{array}$$

$$
\begin{aligned}
&= \left[w\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)-(2\alpha/R)\right]^{-1}\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)\\
&\quad\times\left[\frac{1}{\gamma}w\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)-(2\alpha/R)\right]\\
&= \Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)\frac{\frac{1}{\gamma}w\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)-(2\alpha/R)}{w\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)-(2\alpha/R)}\\
&= \Phi'\left(\widehat{y}_I(\overline{a}^*,\alpha,R,w)\right)\frac{\varepsilon_s w\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)-(2\alpha/R)}{w\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)-(2\alpha/R)}\\
&= \Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)\frac{\frac{1}{\Phi'(\widehat{y}_I(\overline{a}^*,R,w,\alpha))}\frac{\Phi(\widehat{y}_I(\overline{a}^*,R,w,\alpha))}{\widehat{y}_I(\overline{a}^*,R,w,\alpha)}w\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)-(2\alpha/R)}{w\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)-(2\alpha/R)}\\
&= \Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)\frac{w\frac{\Phi(\widehat{y}_I(\overline{a}^*,R,w,\alpha))}{\widehat{y}_I(\overline{a}^*,R,w,\alpha)}-(2\alpha/R)}{w\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)-(2\alpha/R)}\\
&= \Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)\frac{\frac{\Phi(\widehat{y}_I(\overline{a}^*,R,w,\alpha))}{\widehat{y}_I(\overline{a}^*,R,w,\alpha)}-\frac{2\alpha}{Rw}}{\Phi'\left(\widehat{y}_I(\overline{a}^*,R,w,\alpha)\right)-\frac{2\alpha}{Rw}}
\end{aligned}
$$

which is positive for all $\overline{a}^* > 0$ since $(\widehat{y}_I(\overline{a}^*,\alpha,R,w),\Phi(\widehat{y}_I(\overline{a}^*,\alpha,R,w)))$ is the intersection point of the graph of the function $\Phi(y)$ with the straight line $(2\alpha/Rw)\,y+(\overline{a}^*/w)$, and thus by convexity of Φ and $\Phi(0)=0$

$$
\frac{\Phi\left(\widehat{y}_I(\overline{a}^*,\alpha,R,w)\right)}{\widehat{y}_I(\overline{a}^*,\alpha,R,w)} > \frac{2\alpha}{Rw}\ .
$$

Therefore by (53) $\partial\overline{a}^*/\partial w < 0$.

d) Follows from (50) and Proposition 4. ■

For an interpretation of these results observe that the previously ambigious effects (48)-(50) have been determined more precisely by sharpening some assumptions. A decisive rôle play here the shareholders' farsightedness and the elasticity of scale of the firm's technology. If the latter is constant, then by part (c) of the proposition an increase in the wage rate leads to a reduction in the equity base. The higher wage cost induces the firm to reduce its production and thus to diminish its equity base. Things are more complex in the case of a change in the interest rate: part (b) shows that, even if the elasticity of scale is constant, it depends also on the value of this constant whether an increase in the interest rate has a positive or a negative effect on capital. If the technology is close to constant returns, then ε_s is close to one, and the effect is negative. On the contrary, if returns to scale are strongly decreasing, the opposite occurs. Interestingly, these effects are independent of β. Where β comes into play is when we consider the effect of a change in the confidence level α. Then a more farsighted attitude (large β) favours the holding of capital not only directly (part (a)) but also indirectly in that it catalyzes the positive effect of a decrease of the confidence level α (part (d)).

3.6 Moral hazard

A result of the static model was that there is an interval of values of the equity basis a where moral hazard occurs. More precisely, since $c = 0$ we can easily determine this interval now as the set of those values for which $\widehat{y}(a)$ is larger than y^*, the output under unlimited liability. Since $y^{**}(a) > y^*$ for all $a < \overline{a}$ and $\widehat{y}_I(0) < y^*$, there must be a critical value $\underline{a}$ such that $\widehat{y}(a) > y^*$ iff $a \in (\underline{a}, \overline{a})$. That value is given by the condition that $\widehat{y}_I(\underline{a}) = y^*$. Therefore, if the dynamically optimal value of output $\overline{y}^*$ is smaller than y^*, then in the dynamic case there is no moral hazard. This is made precise in the following proposition and illustrated in Figure 3.

Proposition 6 *Assume that* $\beta < 1/R$ *and* $\alpha < \left(3 - \sqrt{5}\right)/2 \approx 0.382$*. Then at the dynamically optimal value of capital* $\overline{a}^*$ *there is no moral hazard.*

Proof. By (16) and (42) $\overline{y}^* < y^*$ iff

$$\left(\Phi'\right)^{-1}\left(\frac{2\alpha}{Rw} + \frac{\beta\left(1-\alpha\right)^3}{w}\right) < \left(\Phi'\right)^{-1}\left(\frac{1}{Rw}\right)$$

which, by strict convexity of Φ, is equivalent to

$$\frac{2\alpha}{Rw} + \frac{\beta\left(1-\alpha\right)^3}{w} < \frac{1}{Rw}$$

$\Leftrightarrow$

$$\beta < \frac{1}{R} \cdot \frac{1-2\alpha}{\left(1-\alpha\right)^3}\,.$$

It is elementary to show that the second factor on the right-hand side is larger than one for $\alpha \in [0,1]$ iff $0 < \alpha < \left(3-\sqrt{5}\right)/2$. This proves the result. ■

Note that, although moral hazard has been overcome, limited liability has been preserved: the cost of bankruptcy is still zero. What the Var-constraint does is *not* to eliminate limited liability; rather, it limits the firm's choice such that the probability of bankruptcy is not larger than α. Moreover, that probability *at the chosen point* $(\overline{a}^*, \overline{y}^*)$ is smaller than it would be under unlimited liability but without the VaR-constraint , i.e. at $(\overline{a}^*, y^*)$. Hence, there is no moral hazard.

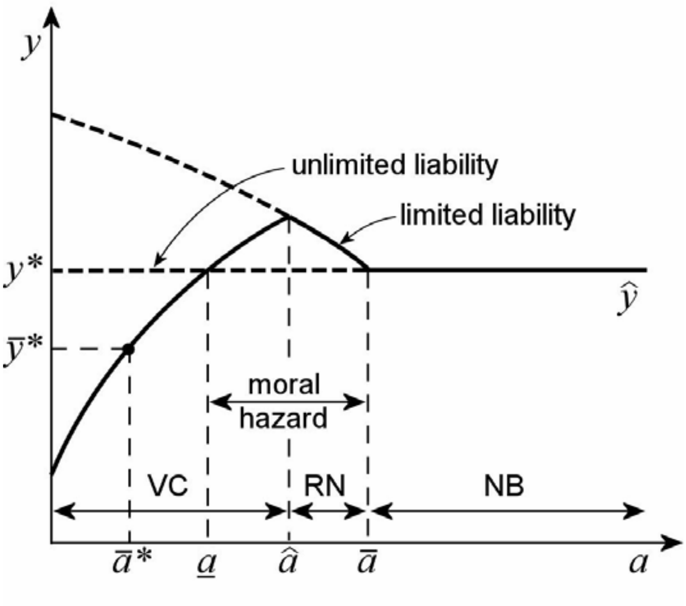

Figure 3: The dynamically optimal choice of the firm and its shareholders.

Example 2 Assume $\Phi(y) = ky^2$. Then by (11)

$$Rwky^2 - 2\alpha y - Ra = 0$$

$\Leftrightarrow$

$$y = \frac{2\alpha \pm \sqrt{4\alpha^2 + 4R^2wka}}{2Rwk} \ .$$

Since $y \geq 0$, this means

$$\widehat{y}_I(a) = \frac{\alpha + \left(\alpha^2 + R^2wka\right)^{1/2}}{Rwk} \tag{54}$$

which further implies

$$\widehat{y}_I'(a) = \frac{\left(\alpha^2 + R^2wka\right)^{-1/2} R^2wk}{2Rwk} = \frac{R}{2}\left(\alpha^2 + R^2wka\right)^{-1/2} \ .$$

Then by (29)

$$v'(a) = (1-\alpha)^2 \frac{R}{2}\left(\alpha^2 + R^2wka\right)^{-1/2}$$

and thus

$$\begin{aligned} v'(a) &= \frac{1}{\beta(1-\alpha)} \\ &\Leftrightarrow (1-\alpha)^3 R\beta = 2\left(\alpha^2 + R^2wka\right)^{1/2} \qquad (55) \\ &\Leftrightarrow (1-\alpha)^6 R^2\beta^2 = 4\left(\alpha^2 + R^2wka\right) \\ &\Leftrightarrow 4R^2wka = (1-\alpha)^6 R^2\beta^2 - 4\alpha^2 \end{aligned}$$

$\Leftrightarrow$

$$\overline{a}^* = \frac{(1-\alpha)^6 R^2\beta^2 - 4\alpha^2}{4R^2wk} \ . \qquad (56)$$

With $\alpha = 0.01$, $\beta = 0.9$, $R = 1.1$ and $w = k = 1$ this yields $\overline{a}^* = 0.190567$ which is smaller than $\underline{a} = 0.198347 < \widehat{a} = 0.202417 < \overline{a} = 0.206612$. The correponding output values are $\overline{y}^* = 0.445725 < y^* = 0.454545 < \widehat{y}_I(\widehat{a}) = 0.45909$. Moreover, using (2) and (8) to calculate the probability of bankruptcy

$$\begin{aligned} \operatorname{prob}(\Pi(y,a) < 0) &= \operatorname{prob}\left(p < \frac{R\left[w\Phi(y) - a\right]}{y}\right) \\ &= \frac{1}{2}\frac{R\left[w\Phi(y) - a\right]}{y} = \frac{1.1\left(y^2 - a\right)}{2y}, \end{aligned}$$

at $(a,y) = (\overline{a}^*, y^*)$ one obtains $0.0194 > 0.01 = \alpha$, the latter being of course the probability of bankruptcy at $(\overline{a}^*, \overline{y}^*)$. Without the *VaR*-constraint but with limited liability, that is at $(\overline{a}^*, y^{**}(\overline{a}^*)) = (0.190567, 0.471271)$ (from (20)), the corresponding number is 0.0368. Thus in case of limited liability, the introduction of the *VaR*-constraint reduces the probability of bankruptcy by $(0.0368 - 0.01)/0.0368 = 72.8\%$ requiring a reduction of $(0.471271 - 0.445725)/0.471271 = 5.4\%$ of output only! These values confirm Proposition 6 and underline quite strikingly the effectiveness of the *VaR*-constraint.

From (56) the signs of $\partial\overline{a}^*/\partial\beta$ and $\partial\overline{a}^*/\partial w$ are obviously consistent with Proposition 5. Regarding $\partial\overline{a}^*/\partial R$, we get

$$\begin{aligned} \frac{\partial\overline{a}^*}{\partial R} &= \left(4R^2wk\right)^{-2}\left\{4R^2wk \cdot 2(1-\alpha)^6 R\beta^2 - \left[(1-\alpha)^6 R^2\beta^2 - 4\alpha^2\right] \cdot 8Rwk\right\} \\ &= \left(4R^2wk\right)^{-2} \cdot 32\alpha^2 Rwk \end{aligned}$$

which is positive whenever $\alpha > 0$. This too is consistent with Proposition 5 (part (b)) as $\varepsilon_s = 1/2 < \overline{\varepsilon}_s$ for $\alpha > 0$.

With regard to $\partial\overline{a}^{*}/\partial\alpha$ note that, from (56) we obtain

$$\overline{a}^{*}>0\Leftrightarrow\beta>\frac{2\alpha}{(1-\alpha)^{3}R}=:\widehat{\beta}$$

which of course is the same condition that we get from (37) if we solve for β and take into account that $\varepsilon_s=1/2$. Hence from (56) we get

$$\frac{\partial\overline{a}^{*}}{\partial\alpha}=\begin{cases}\frac{-6(1-\alpha)^{5}R^{2}\beta^{2}-8\alpha}{4R^{2}wk}<0 & \mathit{if}\ \beta>\widehat{\beta}\\ 0 & \mathit{if}\ \beta<\widehat{\beta}\end{cases}\ .$$

With the usual numbers, $\widehat{\beta}=0.0193$. By (44) $\widehat{\beta}$ is smaller than $\underline{\beta}(\alpha)$ iff

$$\frac{2\alpha}{R(1-\alpha)^{3}}<\frac{2}{3R(1-\alpha)^{2}}\Leftrightarrow\alpha<\frac{1-\alpha}{3}\Leftrightarrow\alpha<1/4\ .$$

Thus, for (see (45)) $\alpha<1-\sqrt{2/3}$ (which is smaller than 1/4), $\beta>\underline{\beta}(\alpha)$ implies $\partial\overline{a}^{*}/\partial\alpha<0$, which is what has been asserted in (d) of Proposition 5.

Turning to $\overline{y}^{*}$, from (54) and (56) we obtain

$$\overline{y}^{*}=\frac{\alpha+\left(\frac{1}{4}(1-\alpha)^{6}R^{2}\beta^{2}\right)^{1/2}}{Rwk}=\frac{\alpha+\frac{1}{2}(1-\alpha)^{3}R\beta}{Rwk}\ .$$

It is immediate that $\partial\overline{y}^{*}/\partial\beta>0$ and $\partial\overline{y}^{*}/\partial w<0$ as asserted in Proposition 4. It is also true that

$$\begin{aligned}\frac{\partial\overline{y}^{*}}{\partial R} &= (Rwk)^{-2}\left[Rwk\cdot\frac{1}{2}(1-\alpha)^{3}\beta-\left(\alpha+\frac{1}{2}(1-\alpha)^{3}R\beta\right)wk\right]\\ &= (Rwk)^{-2}(-\alpha wk)<0\ \mathit{for}\ \alpha>0\ .\end{aligned}$$

Finally, for $\beta>\underline{\beta}(\alpha)$ by (44)

$$\begin{aligned}\frac{\partial\overline{y}^{*}}{\partial\alpha} &= \frac{1-\frac{3}{2}(1-\alpha)^{2}R\beta}{Rwk}<\frac{1-\frac{3}{2}(1-\alpha)^{2}R\underline{\beta}}{Rwk}\\ &= \frac{1-\frac{3}{2}(1-\alpha)^{2}R\frac{2}{3R(1-\alpha)^{2}}}{Rwk}=0\end{aligned}$$

as predicted by (43).

4 Concluding remarks

In this paper we have investigated the behaviour of a firm which is subject to a Value-at-Risk constraint. The rationale for doing this is to discipline the firm in its choices under uncertainty - bearing the risk of bankruptcy - and limited liability. These circumstances create a moral-hazard problem as the firm can shift a part of the cost of risk-taking to its creditors. This may distort its incentives towards behaving in a gambling way. A Value-at-Risk constraint limits this distortion in that it induces the firm to abandon a risk-neutral attitude in case the risk of bankruptcy is about to exceed a certain predefined probability, namely, the confidence level. Thus the firm comes to behave as if it were kind of risk averse, albeit not risk averse in the conventional sense.

In a static set-up, when the capital endowment or equity base of the firm is given, the type of the firm's behaviour - "VaR-constraint risk averse" or risk neutral - varies according to the size of the equity base. The different regimes create a non-monotonicity in the firm's output decision with respect to the capital endowment.

In a dynamic framework capital can be chosen in each period by selecting the corresponding share of dividend payments to shareholders. The model implies that the optimal expected amount of capital to be retained in each period (and thus the expected dividend share) is constant over time and lies in a subset of the regime where the VaR-constraint is binding and where the moral-hazard problem does not arise, even though limited liability is preserved. Thus Value-at-Risk achieves to reconcile two apparently conflicting goals, namely, to encourage entrepreneurial activity by means of limited liability and to avoid irresponsible gambling due to the incentives provided by it.

The comparative-statics analysis of the dynamically optimal values of retained capital and output with respect to the shareholders' time-preference rate, the interest rate, the nominal wage and the confidence level reveals some interesting effects. In particular, with respect to the confidence level, the reaction of the optimal output level is reversed in the case shareholders are farsighted as opposed to the one in which they are shortsighted. Also, it turns out that the elasticity of scale of the firm's production technology plays an important rôle: the reaction of the optimal amount of capital to changes in the interest rate is sensitive to it in that an almost linear technology implies that capital holding increases when the interest rate decreases while with sufficiently strong decreasing returns to scale the opposite occurs.

The basic justification in the current paper for introducing the use of VaR is the possibility of bankruptcy and limited liability in case of its occurrence. Limited liability in turn is at the heart of the functioning of the capitalist system. Tightening or even abolishing it would clearly have a negative effect on economic activity.[8]

The effect of the VaR-constraint on firm behaviour suggests that it might be fruitfully applied to banks as well in the attempt to induce them to adopt a prudent behaviour. Recent papers (Hellman, Murdock and Stiglitz, 2000, Repullo, 2004, Decamps, Rochet and Roger, 2004) have analyzed the rôle and the effectiveness of capital requirements as a way of prudential regulation of risk-taking in banking. The current paper, in having explored the impact of a Value-at-Risk constraint on capital holding by a firm, advocates its potential usefulness for the banking sector as well, particularly as the current credit crunch suggests that it is having difficulties due to over-confident attitudes. Further research along these lines is on the authors' agenda.

Appendix

Lemma 1 $\widehat{y}_I(a)$ *is a strictly increasing function.*

Proof. We analyse the derivative

$$\frac{dy}{da} = -\frac{\frac{\partial F}{\partial a}(y,a)}{\frac{\partial F}{\partial y}(y,a)}.$$

The numerator expresses how prob$\{py - R[w\Phi(y) - a] < 0\}$ varies with a. It is obvious that a higher value of a increases the value on the left-hand side of the inequality. Thus for a fixed density $g(p)$ the probability of satisfying it declines and $\partial F/\partial a$ is negative.

Next consider the function $p = \Psi(y, a, \pi)$ defined in (4). Its partial derivative with respect to y is

$$\begin{aligned}\frac{\partial \Psi}{\partial y} &= \frac{1}{y} Rw\Phi'(y) - \frac{1}{y^2}\left\{\pi + R\left[w\Phi(y) - a\right]\right\} \\ &= \frac{Rw}{y}\left[\Phi'(y) - \frac{\Phi(y)}{y}\right] + \frac{a-\pi}{y^2}.\end{aligned}$$

By convexity of Φ and $\Phi(0) = 0$ this expression is positive for any $\pi \leq 0$. Thus the price realization that yields a fixed $\pi \leq 0$ increases with y. Therefore, for a fixed probability distribution over prices, the probability to

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[8] See e.g. Berkowitz and White (2004) and Fan and White (2003).

realize a non-positive profit increases with y, which is equivalent to saying that $\partial F/\partial y$ is positive. This proves the claim. ■

To show that $y^{**}(a)$ is a strictly decreasing function we shall apply the following

Lemma 2 *Let $h(x,a)$ be a twice differentiable function such that $x^*(a) := \arg\max_x h(x,a)$ exists for all a and*

$$\frac{\partial^2 h(x,a)}{\partial x \partial a} < 0 \tag{57}$$

for all x and a. Then $x^*(a)$ is strictly decreasing.

Proof. Let $a_1 > a_0$ and assume to the contrary that there exists $\overline{x} \geq x^*(a_0)$ such that $h(\overline{x}, a_1) \geq h(x, a_1)$ for all x. Then $h(\overline{x}, a_1) \geq h(x^*(a_0), a_1)$ and

$$\int_{x^*(a_0)}^{\overline{x}} \frac{\partial h}{\partial t}(t, a_1)\, dt = h(\overline{x}, a_1) - h(x^*(a_0), a_1) \geq 0.$$

Thus by (57)

$$\int_{x^*(a_0)}^{\overline{x}} \frac{\partial h}{\partial t}(t, a_0)\, dt > \int_{x^*(a_0)}^{\overline{x}} \frac{\partial h}{\partial t}(t, a_1)\, dt \geq 0$$

and therefore

$$h(\overline{x}, a_0) = \int_{x^*(a_0)}^{\overline{x}} \frac{\partial h}{\partial t}(t, a_0)\, dt + h(x^*(a_0), a_0) > h(x^*(a_0), a_0)$$

which is a contradiction. ■

Lemma 3 *$y^{**}(a)$ is a strictly decreasing function.*

Proof. We seek to sign $\partial^2\Gamma(y,a)/\partial y\partial a$ where $\Gamma(y,a) = \mu(y,a)+\mu_1(y,a)$. By (13) $\partial^2\mu(y,a)/\partial y\partial a = 0$ whereas, by (14),

$$\begin{aligned}\frac{\partial\mu_1(y,a)}{\partial y} &= R^2\left\{-\frac{1}{4y^2}\left[w\Phi(y)-a\right]^2+\frac{1}{2y}\left[w\Phi(y)-a\right]w\Phi'(y)\right\}\\ &= \frac{1}{2y}R^2\left[w\Phi(y)-a\right]\left\{w\Phi'(y)-\frac{1}{2y}\left[w\Phi(y)-a\right]\right\},\end{aligned}$$

and hence

$$\begin{aligned}\frac{\partial^2\mu_1(y,a)}{\partial y\partial a} &= \frac{1}{2y}R^2\left\{-w\Phi'(y)+\frac{1}{2y}\left[w\Phi(y)-a\right]-\left[w\Phi(y)-a\right]\left(-\frac{1}{2y}\right)\right\}\\ &= \frac{1}{2y}R^2\left\{-\frac{a}{y}-w\Phi'(y)+\frac{w\Phi(y)}{y}\right\}<0\end{aligned}$$

by strict convexity of Φ and $\Phi(0) = 0$. This implies $\partial^2\Gamma(y,a)/\partial y\partial a < 0$. Hence we can apply the previous Lemma to conclude that $y^{**}(a)$ is decreasing for $a < \overline{a}$. ■

References

Bauer, C., Value at Risk using hyperbolic distributions, 2000, *Journal of Economics and Business*, Vol. 52, 455-467.

Berkowitz, J. and White, M.J., 2004, Bankruptcy and Small Firms' Access to Credit, *Rand Journal of Economics*, 69-84.

Decamps, J.-P., Rochet, J.-C. and Benoît, R., 2004, The three pillars of Basel II: optimizing the mix, *Journal of Financial Intermediation* 13, 132-155.

Duffie, D. and Pan, J., 1997, An overview of value-at-risk, *Journal of Derivatives*, 4, 7-49.

Fan, W. and White, M.J., 2003, Personal Bankruptcy and the Level of Entrepreneurial Activity, *Journal of Law & Economics*, pp. 543-567.

Greenwald, B.C. and Stiglitz, J.E., 1993, Financial Market Imperfections and Business Cycles, *Quarterly Journal of Economics* 108 (1), 77-114.

Hellman, T.F., Murdock, K.C. and Stiglitz, J.E., 2000, Liberalization, Moral Hazard in Banking, and Prudential Regulation: Are Capital Requirements Enough?, *American Economic Review* 90 (1), 147-165.

Jorion, P., 1997, *Value at Risk: The New Benchmark for Controlling Market Risk, McGraw-Hill*, New York.

Kast, R., Luciano, E. and Peccati, L., 1998, VaR and Optimization, Proceedings of the 2nd International Workshop on Preferences and Decisions, Trento, 1-3 July 1998.

Lüthi, H.-J. and Doege, J., 2005, Convex risk measures for portfolio optimization and concepts of flexibility, *Mathematical Programming*, Series B 104, 541-559.

Myers, S.C. and Majluf, N.S., 1984, Corporate Financing and Investment Decisions when Firms have Information that Investors Do Not Have, *Journal of Financial Economics*, 13, 187-221.

Pritsker, M., 1997, Evaluating Value at Risk Methodologies: Accuracy versus Computational Time, *Journal of Financial Services Research* 12:2/3, 201-242.

Repullo, R., 2004, Capital requirements, market power, and risk-taking in banking, *Journal of Financial Intermediation* 13, 156-182.

Rockafellar, R. T. and Uryasev, S., 2000, Optimization of conditional value-at-risk, *Journal of Risk*, Vol. 2/Number 3, 21-41.

Rockafellar, R. T. and Uryasev, S., 2002, Conditional value-at-risk for general loss distributions, *Journal of Banking & Finance* 26, 1443-1471.

Tulli, V., 2004, *Teoria del Value-at-Risk e struttura finanziaria dell'impresa*, Monografie dell'Istituto di Econometria e Matematica per le applicazioni economiche, finanziarie ed attuariali, Università Cattolica del Sacro Cuore, Milano.

Tulli, V., and Weinrich, G., 2009, A firm's optimizing behaviour under a value-at-risk constraint, *Optimization* 58 (2), 213-226.

Yiu, K.F.C., 2004, Optimal portfolios under a value-at-risk constraint, *Journal of Economic Dynamics and Control* 28 (7), 1317-1334.