Attributing Systemic Risk to Individual Institutions[1]

Methodology and Policy Applications

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Abstract

We propose to measure the systemic importance of individual financial institutions by attributing systemic risk to them on the basis of a construct from cooperative games: Shapley values. The Shapley value methodology provides a general framework for the analysis of different drivers of systemic importance: e.g. the institution's size, individual riskiness and exposure to common risk factors. We prove that, keeping the risk drivers constant across institutions, systemic importance increases faster than size. We also show how the Shapley value methodology can be used for the calibration of policy tools, paying particular attention to the relevance of specific applications of the methodology for alternative policy objectives.

Keywords: Systemic importance, macroprudential approach, Shapley value

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1. Introduction

A key policy lesson from the recent financial crisis has been the need to put greater emphasis on a systemic approach to financial stability. Problems with portfolios of sub-prime mortgages developed into a systemic crisis that engulfed financial institutions and markets across the world, triggering a severe economic recession. The failure of individual institutions helped propagate the shocks across the system. As a result, building better defences against systemic risk has emerged as a policy priority, as has the objective of strengthening the macroprudential orientation of financial stability frameworks.[3]

An operational macroprudential policy framework requires a gauge of the systemic importance of individual institutions. The reason is that key instruments available to policymakers are applied at the firm level. This is true of tools to mitigate *ex ante* the risk of systemic disruptions, such as regulatory minimum capital and liquidity requirements. It is also true of *ex post* supervisory interventions to contain the systemic externalities from distress in specific institutions deemed to be of great systemic importance. A recent example of such an intervention is the emergency financial support provided to AIG at the heat of the crisis.

Measuring systemic importance by attributing system-wide risk to individual institutions is akin to a problem already tackled by game theorists. In his search of a solution to cooperative games, Lloyd Shapley (1953) developed an attribution methodology that carries his name: Shapley value. The portion of the overall value that this methodology attributes to each player in a game equals the average of this player's marginal contributions to the value created by all possible subsets of players. The resulting attribution is fair in the sense that the value created jointly by two players is split equally between them.

In order to measure individual institutions' systemic importance, this paper transposes the Shapley value methodology to the field of risk attribution. In addition to its fairness property – whereby the incremental risk created by the interaction of two institutions is split equally between them – the methodology possesses a number of other desirable features. It is simple, yet efficient in the sense that the shares of systemic risk attributed to individual institutions add up *exactly* to the total. It is flexible since the sufficient conditions for its application are so weak that it can be applied to any measure of risk that treats the system

[3] See BIS (2009), G20 (2009), De Larosiere (2009), FSB (2009). The main distinction between the macro- and microprudential perspectives is that the former focuses on the financial system as a whole, whereas the latter focuses on individual institutions. See Crockett (2000), Knight (2006), and Borio (2003 and 2009) for an elaboration of the macroprudential approach and progress in its implementation.

as a portfolio of institutions. It also encompasses alternative attribution procedures that have been studied in different contexts in the literature. Finally, it can deal with model and parameter uncertainty as it can easily combine information from different risk models and address estimation noise in order to produce robust assessments of systemic importance.

Besides introducing the Shapley value to the field of systemic risk, the paper makes three main contributions. First, the paper demonstrates that the Shapley value methodology provides a general framework for the analysis of different drivers of systemic importance. Concretely, Shapley values underpin a theorem that, when two institutions differ only in terms of size, the ratio of the systemic importance of the bigger institution to that of the smaller institution is larger than the ratio of the respective sizes. The policy implication of this result is that prudential penalties for systemic importance should increase *faster* than size. In addition, the methodology allows for investigating the relative role of different drivers of systemic importance – besides an institution's size, its probability of default (PD) and strength of exposure to a common risk factor – in an empirical setting.[4] On the basis of Moody's KMV data and estimates of risk parameters, we find that size is the principal determinant of systemic importance in a system of 60 large banks.

A second contribution of the paper is to illustrate implications of different policy interventions that target financial stability. The three interventions we consider impose capital requirements at the level of individual institutions and share one objective: a particular level of risk at the level of the overall system. The first intervention attains this objective while equalising the PD of individual institutions, which is in the spirit of the current regulatory framework. The second intervention attains the same level of systemic risk but equalises a macro characteristic across institutions: the levels of their systemic importance. Finally, the third intervention minimises aggregate capital holdings, given the target level of systemic risk. An interesting robust result is that, when institutions differ only with respect to their exposures to a common risk factor, the capital charges that equalise the levels of systemic importance across institutions are associated with a lower level of aggregate capital than the charges that equalise individual PDs.

As a third contribution, the paper identifies and analyses two different applications of the Shapley value methodology in the context of systemic *tail* risk. We argue that the two

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[4] Throughout the paper there is a distinction between the terms systemic (or system-wide) risk and systematic (or common) risk. The former refers to the risk that problems will arise that will impede the ability of the financial system to function. The latter refers to the commonality in risk exposures of financial institutions (in the same spirit as the "market" is analysed in the CAPM). This means that systemic risk can have systematic and idiosyncratic components.

procedures, which have been studied separately in the literature, embed different notions of systemic importance and, thus, should be used for different policy objectives. While one of the procedures measures *contributions* to systemic risk, the other one measures *participation* in systemic events. The first procedure defines tail events at the level of each subsystem, which allows it to capture the extent to which the interaction of different institutions creates risk and, ultimately, the contribution of each institution to the cost of systemic events. The procedure is thus suited for the calibration of macroprudential tools that are designed to influence this contribution. By contrast, the second procedure defines tail events only at the level of the entire system and measures institutions' participation in these events. This makes it the procedure to use in deriving actuarially fair premia for insurance against systemic events.

This paper considers only some of the issues that will need to be tackled by an authority with a systemic perspective on financial stability. We propose an allocation procedure that can be applied to a wide spectrum of systemic risk models and metrics but do not advocate any such model or metric because the optimal ones would depend on the policy context. In our numerical analysis, we employ a highly stylised model and use specific metrics of systemic tail risk: value-at-risk and expected shortfall. These are only for illustration purposes as most results of the analysis do not hinge on our choice of a model or a risk metric. An exception is the result on the relationship between size and systemic importance, which is derived under weak model requirements but only in the context of expected shortfall.

The rest of this paper is organized as follows. Section 2 reviews existing methods for the measurement of systemic risk and the attribution of this risk to individual institutions. Section 3 presents the Shapley value approach to gauging systemic importance and identifies two specific applications of the methodology under standard measures of systemic tail risk. Section 4 proves a general relationship between the size of an institution and its Shapley value. In order to facilitate the analysis in subsequent sections, Section 5 defines a concrete probability distribution of losses in the system and then outlines how this distribution and the associated Shapley values are derived numerically. Section 6 analyses differences between the two specific applications of the Shapley value methodology at a conceptual level. Then, Section 7 studies these differences in an empirical setting. This section also evaluates the relative impact of alternative drivers of systemic importance. Finally, Section 8 provides examples of how the attribution of systemic risk can be used in a policy setting.

2. Related literature

The related literature can be divided into two streams. The first focuses on measuring total system-wide risk when the system is considered as a portfolio of institutions. The second stream studies procedures for attributing total system-wide risk to individual institutions. A key contribution of our paper is to propose a general attribution methodology that (i) can be applied to any systemic risk measures developed in the first stream of the literature and (ii) subsumes as special cases all attribution procedures studied in the second stream.

2.1 Measuring overall risk: from investment portfolios to financial systems

The literature has developed several measures of systemic risk. Of particular interest are those that treat explicitly the financial system as a portfolio of institutions. Examples include the measures used in Geluk et al (2009), Kuritzkes et al (2005), BIS (2008, 2009), Goodhart and Segoviano (2008), and IMF (2008, 2009). In the context of the methodology developed in this paper, these measures of systemic risk are relevant for two reasons. First, they all provide a *single* metric of systemic risk that encompasses *all* institutions in the system. Second, they can be applied to *any* subset of institutions in the system. Given these two features, the quantum or systemic risk implied by a measure can be allocated across institutions on the basis of the attribution methodology we present below in Section 3.

2.2 Attributing risk

An attribution method decomposes the aggregate quantum of risk in order to allocate it across individual contributors. Even though a number of such methods have been discussed in the literature, they have been applied mostly in the context of investment portfolios. As pointed out by Acharya and Richardson (2009), however, the close correspondence between measures of portfolio risk and measures of systemic risk leads naturally to a correspondence between the respective attribution methods. In this section, we discuss attribution methods that have been applied to either of the two types of risk measures.

The most popular method for allocating risk across individual investment exposures considers the losses each one of them is expected to generate in an event of general distress (Praschnik et al (2001), Hallerbach (2002), Kurth and Tasche (2003) and Glasserman (2005)). The method has been recently used by Acharya et al (2009) to obtain indirect measures of the systemic importance of financial institutions. It is also used by Huang et al (2009) in the context of Asia-Pacific banks. An appealing feature of this method is that the portions of risk it attributes to different exposures add up *exactly* to the chosen measure of portfolio risk. However, the method cannot be applied to cases where system-wide risk is not measured by reference to a particular set of events, as would be the case when the choice of risk metric is the variance or higher moments of the portfolio (or system-wide) loss distribution. We show below that this attribution method is a specific application of the Shapley value methodology.

Koyluoglu and Stoker (2002) decompose the variance of losses on an investment portfolio using several approaches, one of which is based on the Shapley value. This, alternative, application of the Shapley value averages the contributions of an exposure to the volatility of the losses on all sub-portfolios to which this exposure belongs. In contrast to the analysis below, Koyluoglu and Stoker (2002) do not consider a measure of general distress, do not derive concrete properties of drivers of Shapley values, do not study alternative applications of the Shapley value methodology and do not examine these applications in a policy context.

A rather different approach underpins CoVaR, which has been applied by Adrian and Brunnermeier (2008) to the market risk of an investment portfolio and suggested as a way to measure the systemic importance of institutions. Applied to a financial system, CoVaR would gauge the severity of distress in one institution, conditional on distress in another institution or in a group of institutions. For example, a CoVaR measure could equal the VaR of bank A losses, conditional on the losses in the entire banking system being equal to their VaR level. Since CoVaR captures the tail interdependence between losses on bank A and those on the banking system, it is a specific measure of the systemic importance of bank A.

That said, the approach embedded in CoVaR and the one we take in this paper are fundamentally different. In this paper, we adopt a top-down approach that gauges systemic importance by attributing system-wide risk to individual institutions. By contrast, CoVaR focuses directly on individual institutions (or groups of institutions) and does not attempt to decompose a measure of system-wide risk. It is a bottom-up approach that does *not* deliver components that add up to the total. In terms of the above example, adding the CoVaRs of all the banks in a system will not deliver the system-wide VaR.

3. Systemic risk and systemic importance

This section, which lays out the analytic foundations of the analysis, is divided in three subsections. The first subsection presents the generic Shapley value methodology as a tool for attributing systemic risk to individual institutions. Then, the second subsection defines two popular measures of risk – value-at-risk and expected shortfall – on the basis of which we analyse Shapley values. Finally, the third subsection considers two concrete attribution procedures that arise as particular applications of the Shapley value methodology under each of the adopted risk measures.

3.1 The Shapley value approach: a general attribution methodology

The Shapley value methodology was developed in the context of cooperative games, in which the collective effort of a group of players generates a shared "value" (e.g. wealth) for the group as a whole.[5] Given such a value, the methodology decomposes it in order to allocate it across players according to their individual contributions. The share of the aggregate value attributed to a particular player is this player's Shapley value.

The Shapley value methodology can be applied directly to a financial system. In this context, the players are institutions and the shared "value," generated by their interrelated activities, is systemic risk. By attributing systemic risk to individual institutions, Shapley values provide a measure of each institution's systemic importance.

For the sake of concreteness, let the financial system be a set N of n institutions (henceforth, "banks"), indexed by $i \in \{1,2,\cdots,n\}$. Let there also be a measure of systemic risk, two alternatives for which we specify in the next subsection. The Shapley values are underpinned by a so-called characteristic function, denoted by ϑ, which satisfies two conditions. First, it is defined on each of the 2^n subsystems of banks, $N^{sub} \subseteq N$.[6] Second, when applied to the entire system, ϑ coincides with the chosen measure of systemic risk.

Given ϑ, the derivation of Shapley values involves the following thought process. Suppose that banks are ordered at random and the subsystem N^{sub} comprises all the banks in front of bank *i* as well as bank *i.* The contribution of bank *i* to the risk of N^{sub} equals the difference between the risk of N^{sub} and the risk of this subsystem when bank *i* is excluded from it: $\vartheta(N^{sub}) - \vartheta(N^{sub} - \{i\})$. The Shapley value of bank *i*, henceforth ShV_i, equals the expected value of such a contribution when the *n!* possible orderings occur with an equal probability. Since the same subsystem may occur at the top of different orderings one obtains:

$$ShV_i(N) = \frac{1}{n} \sum_{n_S=1}^{n} \frac{1}{c(n_s)} \sum_{\substack{N^{sub} \supset i \\ |N^{sub}|=n_S}} \left(\vartheta(N^{sub}) - \vartheta(N^{sub} - \{i\}) \right) \tag{1}$$

In this expression, $N^{sub} \supset i$ are all the subsystems $N^{sub} \subseteq N$ containing bank *i*, $|N^{sub}|$ stands for the number of banks in subsystem N^{sub}, and $c(n_s) = (n-1)!/(n-n_s)!(n_s-1)!$ is the number of subsystems that contain bank *i* and are comprised of n_s banks. In addition, the empty set carries no risk: $\vartheta(\emptyset) = 0$.

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5 The discussion of Shapley value in this paper draws heavily on Mas-Colell et al (1995), pages 679-684. The Shapley value was first introduced in Shapley (1953).

6 These subsystems are: *Ø, {1}, {2}, {3}, …, {n}, {1,2}, {1,3}, …, {n-1,n}, …, {1,2,3,…,n}*.

For a given characteristic function ϑ, the Shapley values of individual banks form a *unique* set of measures of systemic importance. Key properties of the Shapley values are:[7]

Additivity (or efficiency): The sum of Shapley values equals the aggregate measure of systemic risk: $\sum_{i=1}^{n} ShV_i\left(N\right) = \vartheta\left(N\right)$.

Linearity of characteristic functions: Let there be a set of *K* characteristic functions $\left\{\vartheta^k\right\}_{k=1}^{K}$ that give rise to a set of *K* Shapley values for a generic bank *i*, $\left\{ShV_i\left(N, \vartheta^k\right)\right\}_{k=1}^{K}$. Then, the composite function $\vartheta = \sum_{k=1}^{K} \alpha_k \cdot \vartheta^k$ gives rise to $ShV_i\left(N, \vartheta\right) = \sum_{k=1}^{K} \alpha_k \cdot ShV_i\left(N, \vartheta^k\right)$. When the weights $\left\{\alpha_k\right\}_{k=1}^{K}$ are probabilities associated with *K* models (parameters) underpinning $\left\{\vartheta^k\right\}_{k=1}^{K}$, the composite Shapley value incorporates model (parameter) uncertainty.

The Shapley value methodology also satisfies an intuitive *fairness* criterion. Namely, the decomposition is such that the incremental amount of systemic risk generated by the simultaneous presence of any two institutions in the system is split equally between them. A specific implication, illustrated in MasCollel et al (1995), is that the increment of the Shapley value of institution *i* caused by the presence of institution *k* equals the increment of the Shapley value of institution *k* caused by the presence of institution *i*. Moreover, this is true for any subgroup of institutions in the system:

$$ShV_i\left(N^{sub}\right) - ShV_i\left(N^{sub} - \left\{k\right\}\right) = ShV_k\left(N^{sub}\right) - ShV_k\left(N^{sub} - \left\{i\right\}\right) \qquad (2)$$

for all *i* and *k*; and all $N^{\text{sub}} \subseteq N$, such that $i, k \in N^{sub}$.

Besides its intuitive appeal, the property in expression (2) helps bring to the fore differences between alternative applications of the general Shapley value methodology. We develop this point in subsection 3.3 below.

3.2 Concrete measures of systemic risk

We study Shapley values on the basis of measures of systemic *tail* risk, MTR. By MTR we understand a measure that is defined on *each* subsystem $N^{sub} \subseteq N$ and is equal to the expectation of aggregate losses in N^{sub}, conditional on a set of tail events $e^{MTR}\left(N^{sub}\right)$:

[7] In addition to the two properties listed in the main text, a Shapley value: (i) is not affected by the labeling of banks (often referred to as a “symmetry property); (ii) equals zero if it refers to a bank carrying no risk (“dummy axiom”).

$$MTR\left(N^{sub}\right) = E\left(\sum_{i \in N^{sub}} L_i \mid e^{MTR}\left(N^{sub}\right)\right) \quad (3)$$

where L_i is a stochastic loss associated with bank *i*. An MTR incorporates the joint probability distribution of $\{L_i\}_{i \in N^{sub}}$, which we assume for now to be simply well-defined and specify exactly in Section 5.1 below.

The two concrete MTR we study are value-at-risk (VaR) and expected shortfall (ES). Each of these measures defines the tail events via a quantile of the distribution of aggregate losses. Denoting these quantiles by q^{VaR} or q^{ES}, respectively:

$VaR\left(N^{sub}; q^{VaR}\right) \equiv \inf\left\{x : \Pr\left(\sum_{i \in N^{sub}} L_i \le x\right) \ge q^{VaR}\right\}$, implying that the tail events under VaR occur when aggregate losses equal the q^{VaR} quantile of their distribution. In turn, a tail event under ES occurs when the losses are equal to or larger than the q^{ES} quantile:

$$ES\left(N^{sub}; q^{ES}\right) \equiv \frac{1}{1-q^{ES}}\left(\begin{array}{l} E\left[\left(\sum_{i \in N^{sub}} L_i\right) \cdot I_{\sum_{i \in N^{sub}} L_i \ge VaR\left(N^{sub}; q^{ES}\right)}\right] \\ + VaR\left(N^{sub}; q^{ES}\right) \cdot \left(q^{ES} - \Pr\left(\sum_{i \in N^{sub}} L_i \le VaR\left(N^{sub}; q^{ES}\right)\right)\right) \end{array}\right)$$

, where *I* is one if a tail event materialises and zero otherwise. Note that the second summand is relevant when the loss distribution is discrete, which is the case in the numerical examples below.

We are agnostic as to whether VaR or ES is the appropriate measure of systemic tail risk. VaR conveys the maximum level of losses exceeded with a given probability but provides little information about the severity of losses above this level. Being a summary statistic (the mean) of tail losses, ES provides more information.[8] However, ES is typically estimated with substantial noise on the basis of real-world data. Estimation noise is smaller in the case of VaR because it refers to a quantile, as opposed to a mean (Heyde et al (2006)).

3.3 Alternative ways to measure systemic importance

For a measure of systemic tail risk, as defined in expression (3), the Shapley values of individual institutions can be based on either of two different characteristic functions. The two characteristic functions, which differ in their treatment of tail events, decompose the same magnitude of systemic risk in different ways. The two subsections below outline in turn the two attribution procedures.

[8] In addition, the so-called “sub-additivity” property may be violated by VaR but not by ES (see Hull (2006)).

3.3.1 Varying tail events: Procedure 1

For any subsystem $N^{sub} \subseteq N$, the characteristic function of Procedure 1, ϑ^{I}, defines tail events at the level of each subsystem and equals

$\vartheta^{\mathrm{I}}\left(N^{sub}\right) = E\left(\sum_{i \in N^{sub}} L_i \mid e^{MTR}\left(N^{sub}\right)\right) = MTR\left(N^{sub}\right)$, where MTR stands for VaR or ES.[9]

Expression (1) then implies that Shapley values under Procedure 1 reflect the *contribution* of individual banks to the severity of systemic events. This is because the procedure gauges the systemic importance of a bank by combining the risk it generates on its own with its contributions to the risks of all possible groups of other banks in the system.

It is useful to consider the characteristic function ϑ^{I} in the light of he fairness property in expression (2). By defining tail events at the level of a subsystem, ϑ^{I} reflects the extent to which the joint presence of two banks *i* and *k* raises the risk in this subsystem. The Shapley value methodology then splits the incremental amount of risk equally between the two banks. Concretely, if the sizes and PDs of two banks *i* and *k* are strictly positive and the measure of tail risk is ES, the following *strict* inequality

$ShV_i\left(N^{sub};\vartheta^{\mathrm{I}}\right) - ShV_i\left(N^{sub} - \{k\};\vartheta^{\mathrm{I}}\right) = ShV_k\left(N^{sub};\vartheta^{\mathrm{I}}\right) - ShV_k\left(N^{sub} - \{i\};\vartheta^{\mathrm{I}}\right) > 0$ holds for all $N^{sub} \subseteq N$, such that $i, k \in N^{sub}$. The inequality is strict for VaR as well, provided that system losses have a continuous distribution.

3.3.2 Fixed tail events: Procedure 2

In contrast to Procedure 1, the characteristic function of Procedure 2, ϑ^{II}, defines systemic events at the level of the entire system and keeps these events constant across subsystems. Concretely, for any subsystem $N^{sub} \subseteq N$,

$\vartheta^{\mathrm{II}}\left(N^{sub}\right) = E\left(\sum_{i \in N^{sub}} L_i \mid e^{MTR}\left(N\right)\right)$. Note that the values of ϑ^{I} and ϑ^{II} differ in general but coincide when the two functions are applied to the entire system, N.

Under Procedure 2, Shapley values reflect the extent to which individual banks *participate* in the system-wide tail events. To see why, note first that $\vartheta^{\mathrm{II}}\left(N^{sub}\right) - \vartheta^{\mathrm{II}}\left(N^{sub} - \{i\}\right) = E\left(L_i \mid e^{MTR}\left(N\right)\right)$, which depends on *i* but not on *S*. Then, by

9 In an empirical analysis of large Canadian banks, Gauthier et al (2010) compare Procedure 1 to alternative measures of systemic importance that are not based on the Shapley value methodology.

expression (1), the Shapley value of bank i is simply the loss it is expected to generate, conditional on the system-wide tail events:

$$ShV_i\left(N^{sub};\vartheta^{II}\right) = ShV_i\left(N;\vartheta^{II}\right) = E\left(L_i \mid e^{MTR}\left(N\right)\right) \text{ for all } i \in N^{sub} \text{ and all } N^{sub} \subseteq N \qquad (4)$$

An important consequence is that the characteristic function ϑ^{II} underpins an application of the Shapley value methodology, in which the fairness property is uninformative:

$ShV_i\left(N^{sub};\vartheta^{II}\right) - ShV_i\left(N^{sub}-\{k\};\vartheta^{II}\right) = ShV_k\left(N^{sub};\vartheta^{II}\right) - ShV_k\left(N^{sub}-\{i\};\vartheta^{II}\right) = 0$ for *each* $i,k \in N^{sub}$ and $N^{sub} \subseteq N$. The first equality in the last expression is simply a special case of the fairness property in expression (2). However, the second equality reveals that, since it takes tail events as given, ϑ^{II} cannot convey how bank k affects the contribution of bank i to the severity of these events and vice versa. More generally, ϑ^{II} does not convey the extent to which the interaction of banks raises systemic risk.

Procedure 2 has been a popular tool for the attribution of the risk of investment portfolios to individual exposures (see Section 2.2 above). However, previous derivations of the procedure have been based on the linearity of the expectations operator, not on the Shapley value methodology. By extension, the properties of Procedure 2 have not been analysed alongside those of Procedure 1.[10] In Section 6 below, we compare the two procedures and argue that they should be used in different contexts.

4. The impact of size on systemic importance

This is the first in a series of sections, in a which we use the Shapley value methodology as a framework for analysing alternative drivers of systemic importance or as a building block of policy tools. The driver of systemic importance we focus on here is the institution's size, i.e. the magnitude of its business that can generate losses for the system. Introducing only a slight – and quite common – restriction in the setup developed in Section 3 above, we characterise the relationship between the size of an institution and its systemic importance and argue that this relationship has an important policy implication.

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[10] A notable exception is Denault (2001), who studies the properties of the two allocation procedures within the set of coherent risk measures (which includes ES but not VaR). Focusing on portfolio risk, he considers an allocation to be fair if it belongs to the "core", i.e. if the portion of overall risk attributed to any subportfolio is not bigger than the risk that the subportfolio would generate on its own. Concerned with the risk in a financial system, we adopt the fairness criterion in expression (2).

Let the losses associated with any bank in the system be proportional to this bank's size. Concretely, let $L_i = s_i \tilde{L}_i$ for all $i \in \{1,2,\cdots,n\}$, where the deterministic s_i is the size of bank *i* and the distribution of the stochastic $\tilde{L}_i$ is independent of s_i. If the measure of systemic risk is ES, we then obtain the following result for any well-defined probability distribution of $\{\tilde{L}_i\}_{i=1}^{n}$:

Theorem: *Let two banks differ only in terms of size: $s_S < s_B$. Suppose further that the contribution of either of these two banks to the ES of any subsystem decreases (weakly) as the number of banks in the subsystem increases. Then, the ratio of the Shapley value of the larger to that of the smaller bank is (weakly) greater than the ratio of the respective sizes:* $\frac{ShV_B}{ShV_S} \geq \frac{s_B}{s_S}$.

The formal statement and proof of the theorem are in the appendix. The proof uses repeatedly the fact that, if the joint failure of the smaller bank with other banks is a tail event, then the joint failure of the larger bank with the same other banks would also be a tail event. However, the converse need not be true. In other words, a larger bank cannot appear in fewer tail events than a smaller bank with an identical risk profile.

The appendix is explicit on how the proof uses the condition on the relationship between contributions to ES and the number of banks in the subsystem. Importantly, we show that this condition is necessary under Procedure 1 but *not* under Procedure 2. In addition, the condition is an intuitive generalisation of the sub-additivity of ES, or that the sum of the ESs of two portfolios is not smaller than the ES of a third portfolio that is the sum of the first two.

The theorem is clearly relevant for prudential policy. In concrete terms, it implies that, all else equal, prudential penalties for systemic importance should increase faster than size. We concretise this point in Section 8 below.

5. Towards a concrete probability distribution of losses

In the rest of our analysis, we refer to numerical examples, which are based on a particular probability distribution of systemic losses. We obtain this distribution and incorporate it into the Shapley value methodology on the basis of: (i) a simple model of stochastic losses; (ii) a data set that we use to calibrate the model; and (iii) numerical algorithms for simulating losses and calculating Shapley vales. We describe each of these components in turn.

5.1 Model of stochastic losses

Let the system we introduced in Section 3 incur losses if and only if one or several of the banks default. Further, let the loss associated with the default of bank *i* be equal to:

$$L_i = s_i \cdot LGD_i \cdot I_i , \quad (5)$$

In this expression, s_i stands for the size of the bank *i* debt (i.e. the book value of its non-equity liabilities). We normalise the overall size of the system, $\sum_{i=1}^{n} s_i = 1$.[11] Loss-given-default, LGD_i, is a known constant that equals the share of s_i lost if bank *i* defaults.[12]

The default indicator variable in expression (5), I_i, is defined as follows. We assume that there is a single period of length τ and bank *i*, which starts with an asset value of V_0, defaults if and only if $V_{i,\tau} < DP_i$, where DP_i is the default point and $DP_i < V_0$. Thus:

$$I_i = 1 \text{ if and only if } V_{i,t+\tau} < DP_i \text{ and } I_i = 0 \text{ otherwise} \quad (6)$$

The value of assets is affected by one risk factor that is common to all banks, M, and another risk factor that is specific to bank *i*, Z_i:

$$V_{i,\tau} = V_0 + \sigma_V\left(\tau, V_0\right)\left(\rho_i \cdot M + \sqrt{1-\rho_i^2} Z_i\right), \text{ for all } i \in \{1,2,\cdots,n\} \quad (7)$$

where the risk factors are mutually independent standard normal variables,[13] $\sigma_V\left(\tau, V_0\right)$ is the volatility of the shock to assets and $\rho_i \in \left[0,1\right]$ is the common-factor loading of the bank.

In this setting, the risk profile of bank *i* is defined by its probability of default:

$$PD_i = \Phi\left(\frac{DP_i - V_0}{\sigma_V\left(\tau, V_0\right)}\right) \quad (8)$$

and its loading on the common factor, ρ_i, which influences the propensity of the bank to default with other banks. For future use, we note that the correlation of the assets of banks *i* and *j* equals $\rho_i \cdot \rho_j$.

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11 This assumption is without loss of generality because, in our analysis, we take the size of banking systems as given and compare only systems of equal sizes. The assumption would have to be relaxed in order to analyse, for example, how an expansion of one or more banks may influence systemic risk by increasing the overall size of the system. A trivial generalisation of the analysis would help address such issues.

12 Note that we abstract from an important aspect of systemic risk stemming from the relationship between the number of defaults and LGD.

13 This assumption circumvents important empirical questions related to the *shape* of probability distributions of asset returns and the associated uncertainty (see, for example, Hull and White (2004) and Tarashev and Zhu (2008)). As discussed above, however, the Shapley value methodology can accommodate such uncertainty.

Expressions (5)-(7) define fully the joint probability distribution of the losses in any subsystem. Traditionally used in analyses of portfolio credit risk, this setup circumvents an important feature of financial systems. Concretely, interrelationships typically arise not only because banks are exposed to common exogenous risk factors but also because of interbank exposures. Such exposures can propagate shocks within the system and create so-called domino effects, implying that the financial system should be considered not only as a portfolio but also as a *network* of institutions.[14] Importantly for the Shapley value methodology, the network would change from one subsystem to another and, thus, the probability distribution of losses associated with a given bank would depend on which other banks are in the subsystem in focus. We abstract from these issues in expressions (5)-(7), which we use to construct numerical examples that illustrate the Shapley value methodology in a parsimonious setting.

5.2 Data and numerical algorithms

We base our numerical exercises on bank balance sheet data and risk parameter estimates provided by Moody's KMV. These statistics relate to 60 large banks at end-2007. Depending on the exercise, we use the statistics either directly (Section 7 below) or in order to pick realistic risk parameters of highly stylised banking systems (Sections 6 and 8).

In deriving the joint probability distribution of losses in the system, $\{L_i\}_{i=1}^n$, we make use of the following three statistics. First, we use the book value of balance-sheet non-equity liabilities (or debt) in order to measure a bank's size. Banks in our sample are of markedly different sizes, as individual debt levels range from 0.1% to 6.4% of the aggregate debt of the 60 banks. Second, we set the time period τ to one year and assume that PD_i equals the one-year expected default frequency (EDF) of bank *i*, as estimated by the Credit Monitor engine of Moody's KMV. EDFs in the sample range between 0.01% and 2.4%, have a mean of 0.4% and a standard deviation of 0.55%. Third, we estimate common-factor loadings, $\{\rho_i\}_{i=1}^{60}$, by applying the factorisation methodology outlined in Tarashev and Zhu (2008) to the correlation matrix of assets returns delivered by the Moody's KMV GCorr model. The obtained common-factor loadings average 0.58 (consistent with a correlation of $0.58^2 = 0.33$) and range between 0.13 and 0.76.[15]

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[14] For an in-depth analysis of the network structure of a national interbank market, see Boss et al (2004).

[15] Pennacchi (2006) argues that risk-neutral probabilities of default, underpinning market prices such as CDS spreads, should be used for deriving systemic risk. The reason is that such probabilities provide useful information about banks' exposures to systematic (or common) risk factors. In line with this argument, the GCorr estimates of asset-return correlations do incorporate information from market prices.

Given these statistics, the only remaining information we need for deriving the probability distribution of losses is banks' LGD. In line with standard practice in the context of stylised models of systemic credit risk, we assume that $LGD_i = LGD = 55\%$ for all i.

Once we have assigned numerical values to the parameters $\{s_i\}_{i=1}^n$, $\{PD_i\}_{i=1}^n$, $\{\rho_i\}_{i=1}^n$ and *LGD*, we derive the probability distribution of the losses $\{L_i\}_{i=1}^n$ via Monte Carlo simulations. In line with expressions (7) and (8), a simulation for bank *i* consists of drawing a standard normal shock and registering a default if and only if this shock is smaller than $\Phi^{-1}(PD_i)$. Importanly, the shocks associated with any two banks *i* and *j* have a correlation of $\rho_i \cdot \rho_j$. Repeating this exercise a large number of times (concretely 1 million), we obtain a distribution of default occurrences. Finally, applying $\{s_i\}_{i=1}^n$ and *LGD* to the distribution of defaults, as indicated by expression (5), we derive the probability distribution of $\{L_i\}_{i=1}^n$.

Having derived the probability distribution of losses, it is straightforward to obtain the VaR or ES of the system or any of its subsystems. First, we obtain the probability distribution of aggregate losses in the (sub)system by summing the simulated losses across the banks in that subsystem. Then, to this distribution, we apply the VaR and ES formulae from Section 3.2. In these formulae, we set $q^{VaR} = 0.999$ and $q^{ES} = 0.998$, respectively.[16]

The computational burden of deriving Shapley values under Procedure 2 is markedly smaller than that under Procedure 1. In the case of Procedure 2, it is sufficient to apply expression (4) to the *n* distributions of bank-level losses. By contrast, Procedure 1 points to expression (1), which requires that the risk measure (VaR or ES) be calculated for 2^n subsystems.

The practical implementation of Procedure 1 then depends on the number of banks in the system. On a Pentium Dual Core 2.66 GHz computer with 4 GB Ram, it takes less than a minute to implement Procedure 1 via expression (1) when there are $n = 5$ banks. However, the computation time for this implementation increases exponentially in *n*. As a result, an implementation of expression (1) is effectively impossible for $n > 30$. That said, when groups of banks share the same size and risk parameters –as in the examples we consider in Sections 6 and 8 below – the effective number of *different* subsystems is smaller than 2^n and the computation time is reduced.

When we work with a system comprising the 60 heterogeneous banks in our data sample, we need to implement Procedure 1 via an approximation to expression (1). Castro et al

[16] The adopted difference between the two quantiles q^{VaR} and q^{ES} renders the values of VaR and ES measures comparable. None of the conclusions in this article hinges on the relative values of q^{VaR} and q^{ES}.

(2009) propose an efficient approximation scheme, which we adopt. Under this scheme, Shapley values are approximated on the basis of random draws (with replacement) from the *n!* possible orderings of banks. On the basis of 1 million Monte Carlo simulations of losses and 10'000 draws of orderings, we estimate the 60 Shapley values under Procedure 1 within 6 hours on the computer described above. In order to evaluate the size of the estimation noise, we run the estimation scheme ten times. Then, we measure discrepancies across runs via the following ratio:

$\frac{1}{60}\sum_{i=1}^{60}\left(\sum_{r=1}^{10}\left|ShV_i^r - \frac{1}{10}\sum_{r=1}^{10}ShV_i^r\right| \Big/ \sum_{r=1}^{10}ShV_i^r\right)$, where i indexes banks and r runs. This ratio turns out to be below 1%, which indicates negligible estimation noise. Importantly for our analysis below, this noise is substantially smaller than the discrepancies between estimates of the Shapley values under Procedure 1 and those under Procedure 2 (see Section 7.1 below).

Even though we do not need to consider asset volatility for the derivation of systemic risk or Shapley values, we need to do so when we show how Shapley values may be incorporated in regulatory capital requirements (see Section 8 below). In that context, we resort to the Moody's KMV estimates of the asset-return volatilities of the 60 banks in our sample. Based on the assumption that $\sigma_V\left(\tau, V_0\right) = \tau\sigma_V V_0$, these estimates average $\sigma_V = 3.5\%$.

6. Comparison between Procedures 1 and 2

In this subsection, we compare the implications of Procedures 1 and 2, in the context of both VaR and ES, and argue that the two procedures should be used in different contexts.

6.1 When the measure of systemic risk is VaR

The differences between the two procedures are seen distinctly under a VaR measure of systemic risk. To illustrate, we consider a stylised hypothetical system that comprises two groups of five banks, with the banks in each group having the same size, PD and common-factor loading. Across groups, banks differ only in terms of size: a bank in the first (second) group accounts for 7% (13%) of the total size of the system (Table 1).

In this system, banks with a positive contribution to systemic risk (captured by Procedure 1) may not participate *at all* in the system-wide tail events (which Procedure 2 focuses on). When default correlations are low (left-hand panel of the table), the system-wide tail events correspond to the failure of two large banks (i.e. a VaR of 14.3 cents on the dollar). Since these events exclude losses from small banks, applying Procedure 2 leads to the conclusion that these banks are of *no* systemic importance. By contrast, Procedure 1 attributes positive systemic importance to all banks. The picture is symmetric when higher default correlations lead to a system-wide VaR (15.4 cents on the dollar) that corresponds to the losses from the failure of four small banks (right-hand panel of Table 1).

Comparison between Procedures 1 and 2: a VaR example

All banks: PD = 0.27% and LGD = 55%

Group A banks: $n_A = 5$; $s_A = 0.07$. Group B banks: $n_B = 5$; $s_B = 0.13$.

	Low default correlation $\rho_A = \rho_B = 0.60$		**High default correlation** $\rho_A = \rho_B = 0.724$	
	Procedure 1	**Procedure 2**	**Procedure 1**	**Procedure 2**
Group A	34.34%	0.0%	28.15%	100%
Group B	65.66%	100%	71.85%	0.0%
total VaR	*14.3* (100%)	*14.3* (100%)	*15.4* (100%)	*15.4* (100%)

Note: Each panel refers to a different banking system. Systemic risk is measured as total VaR at the 99.9% confidence level, in cents per dollar exposure to the system. The first two rows report the overall share of each group of banks in total VaR, as determined by the procedure specified in the column heading. The number of banks in group *j* equals n_j, the size of a bank in group *j* is s_j and the exposure of a bank in group *j* to the common factor is denoted by ρ_j.

Table 1

The two procedures deliver contrasting messages because they capture differently the impact of interactions among institutions on systemic risk. Since it takes tail events as given, Procedure 2 fails to convey the impact of a given bank on the risk generated by other banks. For the system at hand, Procedure 2 fails to convey that the level of systemic risk is partly the result of the *simultaneous* presence of the *two* groups of banks. When default correlations are low, for example, systemic risk drops by a factor of two if the group of small banks are excluded from the system. In order to capture this impact, it is necessary to consider tail events at the level of each subgroup of banks, which is what Procedure 1 does.

Admittedly, a stochastic LGD would alter the numerical results in Table 1. For example, it would dampen the distinction between the two groups of banks under Procedure 2. The reason is that, if the probability distribution of LGD is continuous, losses from each bank will enter the set of systemic events underpinning the VaR at any confidence level. This would guarantee a strictly positive level of systemic importance for each bank under Procedure 2.

That said, two points should be kept in mind. First, a continuous distribution of LGD would lead to a continuous loss distribution. As discussed in Hallerbach (2002), this would give rise to significant computational issues when Procedure 2 is applied to a VaR measure of systemic risk. Second, keeping such issues aside, stochastic LGD does not alter the fact that, since Procedure 2 takes systemic events as given, it is not designed to convey the degree to which the interaction among banks raises the severity of these events. Numerical results, available upon request, reveal that the differences between the implications of Procedures 1 and 2 remain material even for a stochastic LGD with a substantial variance.

6.1 When the measure of systemic risk is ES

Under the ES measure, it would also be the case that a bank's contribution to system-wide risk would not match this bank's participation in the corresponding system-wide tail events. In order to illustrate this, we consider a hypothetical system comprised of four banks that are of the same size but differ with respect to their PDs and loadings on the common risk factor. In order to analyse differences between Procedures 1 and 2, it suffices to consider the bank with the highest and that with the lowest probability of default, dubbed C and D, respectively. In addition, Bank C features the lowest exposure to the common factor, whereas bank D features the highest exposure (Table 2).

Comparison between Procedures 1 and 2: an ES example

All banks: s = 0.25 and LGD = 55%

	$PD_A = PD_B = 0.31\%$, $PD_C = 0.62\%$, $PD_D = 0.28\%$ $\rho_A = \rho_B = 0.65$, $\rho_C = 0.10$, $\rho_D = 0.74$			
	Banks A and B	**Bank C**	**Bank D**	Total ES
Procedure 1	53%	20%	27%	18.4 (100%)
Procedure 2	49%	26%	25%	18.4 (100%)

Note: Each panel refers to a different banking system. Systemic risk is measured as total ES at the 99.8% confidence level, in cents per dollar exposure to the system. The first three rows report the share of each bank (or group of banks) in total ES, as determined by the procedure specified in the column heading. The size of a bank is denoted by $\boldsymbol{s}$, the PD of bank j is PD_j and the exposure of bank j to the common factor is denoted by ρ_j.

Table 2

Procedure 1 attributes a *larger* share of systemic risk to bank D than to bank C. The underlying reason is that, with its greater dependence on the common risk factor, bank D is more likely to be part of joint failures than is bank C. This raises the contribution of bank D to systemic risk relative to that of bank C. For example, removing bank D from the overall system makes the system-wide ES drop from 18.4 to 15.3 cents on the dollar, while removing bank C induces a smaller drop, to 17.6 cents. Procedure 1 incorporates such facts directly by considering the extent to which each bank raises the ESs of various subsystems.

For the same system, Procedure 2 delivers a different conclusion: that the systemic importance of bank D is *smaller* than that of bank C. To see why, note that the system-wide tail events in the considered system correspond to losses generated by the failure of one or more banks. Note also that the conditional expectation in expression (4) captures the degree to which a bank participates in system-wide tail events, but not its propensity to do so *with other banks*. Thus, under Procedure 2, the high likelihood of *solo* failures by bank C in the system-wide tail events drives its Shapley value above that of bank D.

The discussion in this and the previous subsection leads us to conclude that the two allocation procedures should be used for different policy objectives. For example, if the objective is to affect individual *contributions* to the severity of systemic events, then Procedure 1 should be used. By contrast, Procedure 2 should underpin a scheme for insuring against system-wide losses in pre-specified tail events. An actuarially fair premium in such a scheme would equal exactly the extent to which a bank is expected to *participate* in these events. Section 8 below revisits the alternative policy objectives.

7. Empirical analysis of allocation procedures and drivers of systemic importance

In this section, we assume that the 60 banks in our sample form a system and then analyse this system from two different perspectives. First, we examine the extent to which the conceptual differences between the two allocation procedures – discussed in Section 6 – have implications for the measured levels of institutions' systemic importance in an empirical setting. We find that the implications are material. Second, we study the role of size, PD and loading on the common risk factor in determining systemic importance. The results indicate that size is the most important driver of systemic importance in our sample.

7.1 Choice of allocation procedure

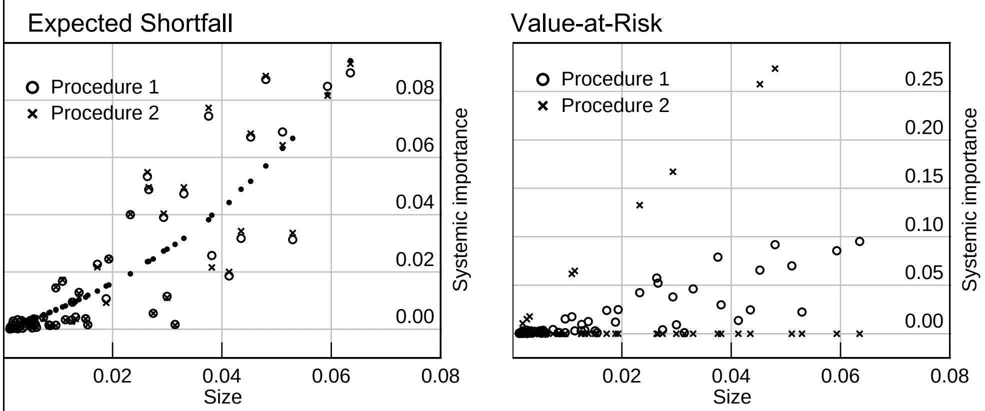
In Graph 1 we plot the levels of systemic importance for the 60 banks, as measured under the ES and VaR measures and under the two allocation procedures. The different definitions of systemic importance incorporated in the two allocation procedures deliver different Shapley values (unfilled circles vs. crosses). When the risk measure is ES, the Shapley values implied by Procedure 2 deviate on average by 10% from those implied by Procedure 1.[17] The deviations are even greater under the VaR measure. In this case, Procedure 2

[17] The deviations between the implications of the two procedures would naturally depend on the financial system in focus. In another, unreported, example, the Shapley values implied by Procedure 2 for a system comprised of the 15 largest Swiss banks deviate on average by 74% from those implied by Procedure 1. Importantly, in the same system, the Shapley values implied by Procedure 2 for UBS and Credit Suisse (the two largest Swiss banks) deviate on average by 78% from those implied by Procedure 1.

attributes positive systemic significance only to the 9 banks that happen to belong to the default configurations at the q^{VaR} quantile of the probability distribution of system-wide losses. At the same time, the Shapley values of all banks are positive under Procedure 1. These results are in line with the discussion in Section 6 above and reveal that using a procedure for a wrong purpose would have material implications.

Allocating systemic risk

System comprised of 60 banks



Note: An institution's systemic importance is measured by its Shapley value, as described in Section 3.1. The sum of Shapley values (under either procedure) equals the ES (left-hand panel) or VaR (right-hand panel) of the system. Crosses and unfilled circles plot results obtained on the basis of the data presented in Section 5.2. Filled circles are obtained under Procedure 1 assuming that PDs and common-factor loadings are the same across banks and equal the respective sample averages, while size is as implied by the data.

Graph 1

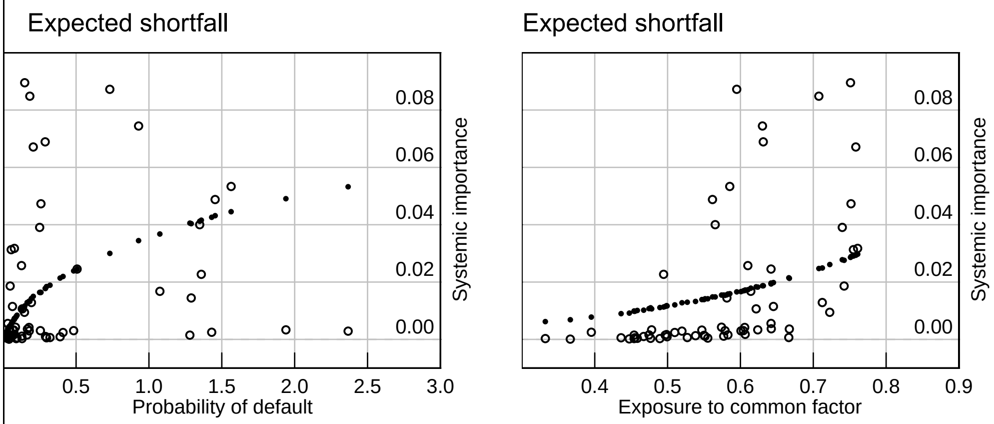
7.2 Drivers of systemic importance

In order to evaluate the relative importance of alternative drivers of systemic importance, we consider two sets of results based on the ES measure and Procedure 1 (Graphs 1, left-hand panel only, and 2). Unfilled circles in the three panels plot respectively the size, PD and the common-factor loading of each of the 60 banks in our sample against the corresponding Shapley values. Filled circles plot the Shapley values that each of the three drivers would generate if the other two drivers were at their sample means. We gauge the importance of a driver by the sum of squared deviations (SSD) between the actual Shapley values (unfilled circles) and the values obtained under the "all else equal" assumption (filled circles). The smaller is this deviation, the greater is the importance of the underlying driver.

Under this criterion, size is the most important driver of the systemic importance of the banks in our sample. When only size is allowed to vary across banks, SSD equals 0.009. In comparison, the corresponding SSDs in the case of common-factor loadings and PDs are substantially higher, at 0.029 and 0.037, respectively.

Alternative drivers of systemic importance

System comprised of 60 banks



Note: See note to Graph 1. Unfilled circles in the left-hand panel of Graph 1 and the two panels of the current graph plot the same measures of systemic importance against different inputs to the calculation of these measures. Filled circles are obtained under Procedure 1 assuming that only bank PDs (left-hand panel) or loadings on the common factor (right-hand panel) are as in the data described in Section 5.2. In each panel, the inputs to the calculation of systemic importance that are different from the one plotted along the horizontal axis are kept fixed at their sample averages. Graph 2

A related result is that, even though higher PDs raise banks' Shapley values when all else is held constant, actual Shapley values exhibit a negative relationship with PDs (Graph 2, left hand panel). This result reflects the stylised fact that bigger institutions tend to feature lower PDs. When size is the principal driver of systemic importance, it then follows that the high systemic importance of larger banks would tend to be associated with lower PDs.

The relationship between size and systemic importance, plotted with filled circles in the left panel of Graph 1, is unsurprising given the Theorem stated in Section 4. Namely, when PD and common-factor exposure are held constant across banks, the graph reveals a convex positive relationship between size and systemic importance. Thus, all else equal, the ratio of the systemic importance of a larger institution to that of a smaller one is greater than these institutions' relative size.

8. Stylised policy interventions

The Shapley value methodology would form a natural basis of prudential policy that operates at the level of individual institutions but has system-wide objectives. In general, such policy would have a direct and indirect impact on risk and the costs of its materialisation. Taking banks' investment positions as given, for example, higher capital buffers would lower directly individual PDs and, thus, systemic risk.[18] In turn, a scheme for insurance against systemic events would have the direct impact of limiting their economic costs. By contrast, an *indirect* policy impact relates to changes in institutions' strategic behaviour. Knowing the relationship between capital requirement or insurance premia and their investment positions, institutions might strategically alter their size, PD or exposure to common risk factors. Both the direct and the indirect impact would affect the vulnerability of the system as a whole. In this paper, we are interested solely in the direct impact of a policy intervention, paying particular attention to the induced changes (if any) in institutions' Shapley values. We leave analysis of the indirect impact to future research.

Suppose that an authority charges actuarially fair premia for insurance against systemic (i.e. tail) events. Abstracting from the behavioural implications of such charges, they can be analysed while keeping banks' risk profiles constant. The argument in Section 6 then implies that the premia should be calibrated on the basis of allocation Procedure 2. If, in addition, the systemic events are defined by an ES measure, the theorem stated in Section 4 implies that, all else equal, the insurance premia should increase faster than banks' sizes.

When an authority wishes to impose capital requirements in order to influence banks' contributions to systemic risk, the situation is different for two reasons. First, capital requirements do have a direct impact on banks' risk profiles. Second, as discussed in Section 6, the objective of the capital requirements implies that they should be set on the basis of allocation Procedure. We now explore these issues in some detail.

We assume that the authority applies capital charges to individual institutions in order to attain a given target for system-wide ES. We discuss three alternative types of policy interventions that attain this target. The first intervention attains the target by imposing capital charges that equalise the PD across all institutions in the system. Centred on a micro feature – bank-level PD – this intervention is in the spirit of the current policy framework.

The other two interventions adopt a macro perspective in attaining the same target for system-wide risk. One of them parallels the first approach but equalises a macro feature across banks: their contributions to system-wide risk.[19] The other one minimises the overall level of capital in the system, i.e. makes sure that the marginal reduction of systemic risk

[18] Prudential buffers could also reflect institutions' funding positions. This, however, is of peripheral importance to the analysis in this paper, which focuses on credit risk.

[19] As argued by Pennacchi (2006), a policy tool that abstracts from an institution's contribution to systemic risk would encourage exposures to systematic risk factors that are higher than their socially optimal levels.

induced by an extra unit of capital would be the same across institutions. Arguably, this last intervention optimises the use of capital at the level of the system.

Our analytical setup is parsimonious. We assume that banks hold only required capital, implying that changes in capital requirements have a direct effect on banks' leverage. At the same time, we assume that capital requirements do not affect either the size or the composition of the asset side of banks' balance sheets. We can then write the following:

$$PD_i = \Phi\left(\frac{\psi \cdot \left(V_0 - K_i\right) - V_0}{\sigma_V V_0}\right) \qquad (9)$$

where K_i is the level of the bank's equity capital and the volatility of its assets equals $\sigma_V\left(\tau = 1, V_0\right) = \sigma_V V_0$. In the light of expression (8), there is an implicit assumption about the default point: $DP_i = \psi \cdot \left(V_0 - K_i\right)$. As initial conditions, we calibrate $K_i/V_0 = 0.04$ and, in order to stay in line with the Moody's KMV data outlined in Section 5.2, we set $\sigma_V = 3.5\%$ and ψ to a value implying a starting PD = 0.3% for all banks. A higher K_i lowers the bank's PD but leaves the level of V_0 and its loading on the common factor, ρ_i, unchanged.

The analysis below considers the three alternative policy interventions in the context of different banking systems. Each of these systems is populated by two groups of banks, with each of the groups accounting for half of the total assets in the system and comprised of homogeneous banks. Banks in different groups differ from each other in terms of specific individual characteristics.

8.1 When banks differ in terms of size

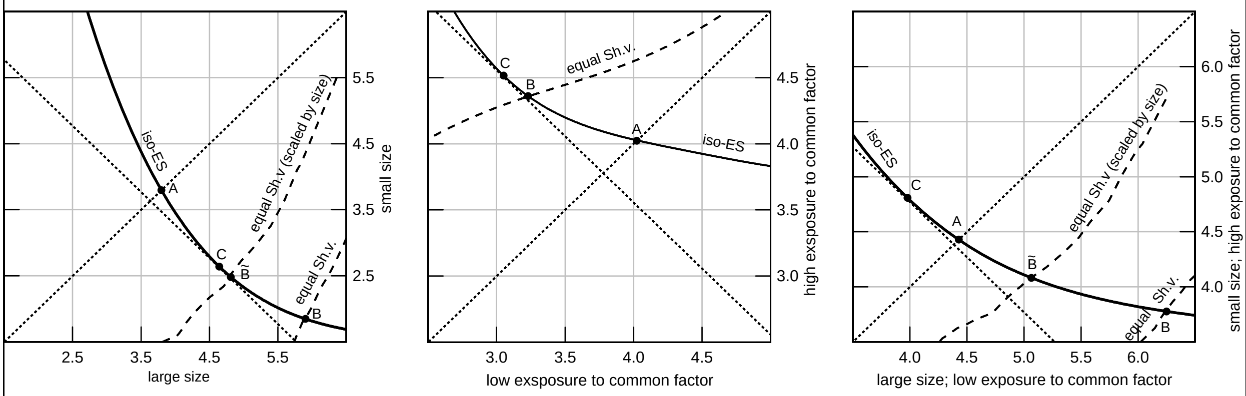
In our first example, banks in different groups differ only in terms of their size. The effects of the alternative policy interventions in this system are depicted in Graph 3 (left-hand panel), where capital charges on large (small) banks are on the horizontal (vertical) axis. The curve labelled *iso-ES* corresponds to the combinations of capital charges that achieve the target level of system-wide risk. The first policy approach attains this target for equal capital charges, at point A: i.e. the intersection between the *iso-ES* curve and the 45-degree line from the origin. By expression (9), equal capital charges are associated with equal PDs. Equal PDs, however, do not lead to equal contributions to systemic risk. At point A, the ratio of the Shapley value of a large bank to that of a small bank is 10.3. In accordance with Theorem 1, this ratio is higher than the corresponding relative size, which equals 4.

Equalising the Shapley values of different banks requires that the PD of a large bank is lower (i.e. its capital charge is higher) than that of a small one. This is illustrated by the dashed curve labelled *equal ShV*. This curve, which is everywhere below the 45-degree line from the origin, plots all capital allocations that result in large banks having the same Shapley values as small banks. The second policy intervention delivers the configuration of capital charges at the intersection between the *equal ShV* and *iso-ES* curves: i.e. point B.

Policy interventions

Capital for two groups of banks

Different banks sizes[1] — Different exposures to common factor[2] — Different sizes and exposures to common factor[3]



Note: Each panel corresponds to a specific system comprising two groups of banks. Risk characteristics are identical across banks in each group. The aggregate group-wide assets are the same across groups. Each axis measures the capital charge for a particular group of banks, as per cent of the group's aggregate assets. The lines labelled "iso-ES" plot the pairs of capital charges, which imply that the system-level expected shortfall equals 10% (left-hand and centre panels) or 8% (right-hand panel) of aggregate liabilities in the system. The two dashed lines plot the capital pairs which imply that Shapley values (respectively, Shapley values scaled by banks' size) are equal across banks.

[1] The system comprises 20 banks that form two homogeneous groups. Banks in different groups differ only in terms of their size, sharing the same loading on the common risk factor: $\rho = 0.50$. The first group comprises 4 large banks, $s_{large} = 0.125$, while the second group comprises 16 small banks, $s_{small} = 0.031$. [2] The system comprises two homogeneous groups of five banks. Banks in different groups differ only in terms of their exposure to the common risk factor: $\rho_{low} = 0.30$ vs. $\rho_{high} = 0.70$. [3] The system comprises 20 banks, divided in two homogeneous groups. The first group comprises 4 large banks with a low exposure to the common factor: $s_{large} = 0.125$, $\rho_{low} = 0.30$. The second group comprises 16 small banks with a high exposure to the common factor: $s_{small} = 0.031$, $\rho_{high} = 0.70$.

Graph 3

An equalisation of Shapley values may be undesirable if the banks differ sufficiently in terms of size. Too low a capital charge for small banks may effectively place them outside of the regulatory perimeter, which could be socially suboptimal for reasons that are not captured by the stylised model in this paper. In turn, capital charges for large banks may be politically unpalatable above a certain level. In such a case, the authority may intervene to equalise Shapley values *scaled by size*, delivering the configuration of capital charges at point $\widetilde{B}$.

The third intervention seeks a capital allocation that minimises the aggregate capital in the system, conditional on the target level of system-wide ES. Graphically, this approach delivers the capital allocation given by the tangency point between the *iso-ES* curve and a straight line with a slope of -1: point C.

8.2 When banks differ in terms of common-factor exposures

In the second example, banks are also homogeneous within a group but now differ across groups only in terms of their exposure to the common risk factor. The capital configurations under each of the three policy interventions are depicted in Graph 3 (centre panel).

In this example, each "macro" intervention (points B or C) leads to efficiency gains in comparison to the "micro" intervention (point A). Concretely, the aggregate level of capital in the system equals 4% of aggregate assets at point A, 3.8% at point B and 3.78% at point C (the minimum).

The relative position of points A and B relates to how common-factor exposures and PDs interact in determining systemic importance. Intuitively, the systemic importance of a bank with a high common-factor exposure is more sensitive to changes in this bank's PD (equivalently, capital charge) than is the systemic importance of a bank with a low common-factor exposure. The flipside of this is that, when banks initially have the same PDs (capital charges) but different common-factor exposures and the goal is to equalise systemic importance across banks, the increase in the capital charge for banks with a greater common-factor exposure is smaller than the corresponding reduction for banks with a lower common-factor exposure. Hence, equalising contributions to systemic risk (point B) leads to a more efficient use of capital in this particular system than achieving the same systemic risk with uniform capital levels (point A).

In turn, the relative positions of points B and C highlights the differences between the two macro interventions. By construction, point C requires the lowest aggregate level of capital. By transferring more capital from the banks with a low systematic factor exposure to the group with a high exposure, this allocation exploits further the scope for efficiency gains than the one that equalises contributions to systemic risk.

The ranking of the first two policy interventions in terms of aggregate capital requirements may differ from the above example if banks in the two groups differ in several aspects. For example, a system where the banks with the higher exposure to the common risk factor are also of a smaller size could lead the first policy intervention to deliver a more efficient use of capital than the second one. This is shown in the right-hand panel of Graph 3.

9. Conclusion

Measures of the systemic importance of financial institutions are key inputs to policy tools with system-wide objectives. In this paper, we have proposed a general and flexible methodology for obtaining such measures and have illustrated how they can be applied in a policy context. We have also used the methodology in order to analyse alternative drivers of systemic importance. The analysis has delivered a theorem that, once risk characteristics have been controlled for, the systemic importance of financial institutions increases faster than their size. We have also demonstrated that different applications of the attribution methodology incorporate different notions of systemic importance and, as a result, should be used for different policy objectives.

Appendix: Formal results on the impact of size on Shapley values

This appendix proves two theorems on the relationship between the size of an institution and its Shapley value. There is one theorem for each of the procedures introduced in Section 3.3. Both theorems consider a general setting, in which the system is a set N of n banks and systemic risk is driven by the losses $\{L_i\}_{i=1}^{n}$. For all $i \in \{1,2,\cdots,n\}$, $L_i = s_i\tilde{L}_i$, where the deterministic s_i denotes the size of bank i and the distribution of the stochastic $\tilde{L}_i$ is independent of $\{s_i\}_{i=1}^{n}$. $\tilde{L}_i$ is henceforth referred to as the "scaled loss" associated with bank i. The measure of systemic risk that the theorems focus on is ES.

The proof of each theorem makes use of the following conditional expectation:

$$E\Big(\sum_{i \in N^{sub}} L_i \mid T\Big) = \sum_{i \in N^{sub}} s_i \cdot \sum_{e \in T} p_e \cdot \tilde{L}_i(e) \cdot I_{i \in e}\text{, where } I_{[i \in e]} = \begin{cases} 1 & \text{if } i \in e \\ 0 & \text{otherwise} \end{cases} \qquad \text{(A.1)}$$

$N^{sub} \subseteq N$, T is a set of tail events, and $e \in T$ is a particular event that occurs with probability p_e. The scaled loss associated with bank i in event e is given by $\tilde{L}_i(e)$. The implicit assumption that probability distributions are discrete is made without loss of generality and for consistency with the exposition in the main text of the paper.

Theorem 1: characteristic function with fixed tail events

Let the system be as described at the beginning of the appendix and the measure of systemic risk be ES. Consider two banks in this system, S and B, which differ in size, $0 < s_S < s_B$, and incur losses with a strictly positive probability. For any subset $G \subset N$ such that $S, B \notin G$, let the joint probability distribution of losses involving S, $\{L_i\}_{i \in G,S}$, be identical to that of losses involving B, $\{L_i\}_{i \in G,B}$. Finally, let Shapley values be based on characteristic function ϑ^{II}, as defined in Section 3.3.2. Then: $\dfrac{s_B}{s_S} \le \dfrac{ShV\,(B)}{ShV\,(S)}$.

Proof of Theorem 1

Note first that the set of relevant tail events T under ϑ^{II} is determined at the level of the whole system. Thus, expressions (4) and (A.1) imply that the Shapley value of bank i equals: $s_i \sum_{e\in T} p_e \cdot \tilde{L}_i(e) \cdot I_{i\in e}$. The ratio of the Shapley values of B and S is then given by:

$$\frac{ShV(B)}{ShV(S)} = \frac{s_B}{s_S} \cdot \frac{\sum_{e\in T} p_e \cdot \tilde{L}_B(e) \cdot 1_{B\in e}}{\sum_{e\in T} p_e \cdot \tilde{L}_S(e) \cdot 1_{S\in e}} \geq \frac{s_B}{s_S}.$$

The reason for the inequality is the following. For each tail event, $e \in T$, that includes S but not B, there must be a corresponding event in T that includes B but not S and has the same probability of occurrence as the former event. This follows from the definition of T under ES, the size difference, $s_S < s_B$, and the assumption that S and B have identical risk characteristics. However, since $s_S < s_B$, it is possible that: (i) there are tail events that include B but not S and (ii) there is no corresponding event that includes S but not B. This implies that $\sum_{e\in T} p_e \cdot \tilde{L}_B(e) \cdot 1_{B\in e} \geq \sum_{e\in T} p_e \cdot \tilde{L}_S(e) \cdot 1_{S\in e}$, which establishes the above inequality and completes the proof of the theorem. ■

Theorem 2: characteristic function with varying tail events

Let the measure of systemic risk, the banking system and banks S and B in it be as specified in Theorem 1. If Shapley values are based on characteristic function ϑ^{I}*, as defined in Section 3.3.1, then the following is a sufficient condition for* $\frac{ShV(B)}{ShV(S)} \geq \frac{s_B}{s_S}$*:*

$$ES(\{i,G\}) - ES(G) \geq ES(\{i,j,G\}) - ES(\{j,G\}),\ \textit{where}\ i,j \in \{S,B\},\ G \subset N\ \textit{and}\ S,B \notin G \quad \text{(A.2)}$$

This condition states that the marginal contribution of bank i to the ES of a subgroup should not decrease as the number of other banks in this subgroup increases. The condition is intuitive because, as the number of banks in the subgroup increases, idiosyncratic risk is diversified away and the impact of each individual bank on the (average) severity of tail events should be expected to decrease. Note that the condition could be rewritten as $ES(\{i,G\}) + ES(\{j,G\}) \geq ES(\{i,j,G\}) + ES(G)$, which collapses to the sub-additivity property of ES when the set G is empty.

Proof of Theorem 2

Note first that, under characteristic function ϑ^{I} , tail events are determined at the level of each subsystem. Note next that, each set of banks G that does not include S and B, there is exactly one pair of marginal contributions $\left(ES(\{S,B,G\}) - ES(\{S,G\})\right)$ and $ES(\{B,G\}) - ES(G)$ that enter the Shapley value of bank *B*. Symmetrically, there is a corresponding pair of marginal contributions for the Shapley value of bank *S*. Expression (1) then implies that

$$\frac{ShV\ (B)}{ShV\ (S)} = \frac{\sum_{G\in\Gamma}\omega(G)\left(ES(\{B,G\}) - ES(G)\right) + \sum_{G\in\Gamma}\tilde{\omega}(G)\left(ES(\{S,B,G\}) - ES(\{S,G\})\right)}{\sum_{G\in\Gamma}\omega(G)\left(ES(\{S,G\}) - ES(G)\right) + \sum_{G\in\Gamma}\tilde{\omega}(G)\left(ES(\{S,B,G\}) - ES(\{B,G\})\right)},$$ where Γ is the collection of all sets *G* that do not contain *S* or *B*, and the weights $\omega(G)$ and $\tilde{\omega}(G)$ change with the number of banks in *G*. In general, $\omega(G) \neq \tilde{\omega}(G)$ because these weights are associated with subsets of the system comprising different numbers of banks.

A simple rearrangement of summands allows to rewrite the last equality as follows:

$$\frac{ShV\ (B)}{ShV\ (S)} = \frac{\Psi^B + \Xi}{\Psi^S + \Xi} \equiv$$

$$\equiv \frac{\sum_{G\in\Gamma}\left(\omega(G) + \tilde{\omega}(G)\right)\left(ES(\{B,G\}) - ES(G)\right) + \sum_{G\in\Gamma}\tilde{\omega}(G)\left\{\left[ES(\{S,B,G\}) - ES(\{S,G\})\right] - \left[ES(\{B,G\}) - ES(G)\right]\right\}}{\sum_{G\in\Gamma}\left(\omega(G) + \tilde{\omega}(G)\right)\left(ES(\{S,G\}) - ES(G)\right) + \sum_{G\in\Gamma}\tilde{\omega}(G)\left\{\left[ES(\{S,B,G\}) - ES(\{B,G\})\right] - \left[ES(\{S,G\}) - ES(G)\right]\right\}}$$

Lemma 1, which is stated and proved below, implies that $\frac{\Psi^B}{\Psi^S} \geq \frac{s^B}{s^S}$. In turn, condition (A.2) implies that $\Xi \leq 0$. Then, since $ShV\ (S) = \Psi^S + \Xi > 0$ and $s_S < s_B$, it follows that

$$\frac{ShV\ (B)}{ShV\ (S)} - \frac{s^B}{s^S} = \frac{\left(\Psi^B - \Psi^S\frac{s^B}{s^S}\right) - \left(\frac{s^B}{s^S} - 1\right)\Xi}{\left(\Psi^S + \Xi\right)} \geq 0$$. This proves the theorem. ■

Lemma 1

Let banks *S* and *B* and the set of banks *G* be as specified in Theorem 2. Then $\frac{\Psi^B}{\Psi^S} \equiv \frac{ES(\{B,G\}) - ES(\{G\})}{ES(\{S,G\}) - ES(\{G\})} \geq \frac{s_B}{s_S}$.

Proof of Lemma 1

Let $T(N^{sub})$ denote the set of tail events underpinning the ES of any set of banks $N^{sub} \subseteq N$. Given (A.1), $ES(G) = k \bullet s_G$, where k is a $1\times m$ vector of probability-weighted scaled losses associated with each of the *m* banks in *G* and the set of tail events *T(G)*, and s_G is the $m\times 1$ vector of respective bank sizes. Similarly, we can express $ES(\{B,G\}) = ts_B + w \bullet s_G$ and $ES(\{S,G\}) = \hat{t}s_S + \hat{w} \bullet s_G$, where t and $\hat{t}$ are scalars and w and $\hat{w}$ are $1\times m$ vectors.

With this notation, the weak inequality in the statement of the Lemma can be expressed as a condition that the sign of the following expression is weakly positive:

$$\frac{ts_B + w \bullet s_G - k \bullet s_G}{\hat{t}s_B + \hat{w} \bullet s_G - k \bullet s_G} - \frac{s_B}{s_S} = \frac{\left(t - \hat{t}\right)s_B s_S + \left(s_B - s_S\right)k \bullet s_G - \left(s_B \hat{w} \bullet s_G - s_S w \bullet s_G\right)}{s_S\left(\hat{t}s_B + \hat{w} \bullet s_G - k \bullet s_G\right)} \tag{A.3}$$

Given that the size of bank S is positive and it incurs losses with a positive probability, it has strictly positive marginal contributions to the ES of the banks in any set *G*, implying that the denominator in (A.3) is positive. Thus, it remains to prove that the numerator is (weakly) positive. We note the following fact:

Fact 1: $w \bullet s_G \leq \hat{w} \bullet s_G$. In other words, the portion of the ES that is attributed to failures of banks in *G* is smaller in the case of set {*B,G*} than in that of {*S,G*}.

To establish the fact, start with reasoning used for the proof of Theorem 1. Each tail event that is in the set *T({S,G}*) and includes bank *S* (and possibly banks in *G*) is matched by a corresponding tail event, in *T({B,G}*), in which *B* replaces *S*. However, the opposite need not be true. Concretely, it cannot be ruled out that a set of tail events $\lfloor B, g \rfloor \in \mathrm{T(\{B,G\})}$ – where $g \in \mathrm{G}$ and $\lfloor B, g \rfloor$ denotes a tail event defined by losses from bank *B* and banks in *g* – is matched by a set of tail events $\lfloor \tilde{g} \rfloor \in \mathrm{T(\{S,G\})}$, where $\tilde{g} \in \mathrm{G}$ and, thus, $S \notin \mathrm{G}$.

Then, establish two properties of tail events $\lfloor \tilde{g} \rfloor$. First, the probability mass of $\lfloor \tilde{g} \rfloor$, conditional on *T({S,G}*), is equal to the probability mass of the "replacement" tail events $\lfloor B, g \rfloor$, conditional *T{B,G}*. This follows from the fact that, since *T({S,G}*) and *T({B,G}*) are defined by the same ES measure, they have the same unconditional probability mass. Second, the portion of the aggregate loss in tail events $\lfloor B, g \rfloor$ that is associated with banks in set g is at most as large as the aggregate loss in tail events $\lfloor \tilde{g} \rfloor$. To see why, note that, by the definition of ES in Section 3.2, the loss associated with any event in $\lfloor \tilde{g} \rfloor$ has to be greater than the loss associated with any event that is not in the set of tail events $T\{S, G\}$. Again by that expression and owing to the identical risk characteristics of banks *S* and *B*, $\lfloor \tilde{g} \rfloor \in T\{S, G\}$ but scaling losses associated with *B* in $\lfloor B, g \rfloor$ by $\frac{s_S}{s_B}$ would not qualify as a tail event in $T\{S, G\}$.

The desired result then follows from $s_S > 0$.

The two properties of tail events $\lfloor \tilde{g} \rfloor$ establish Fact 1.

In turn, Fact 1 points to a lower bound on the numerator of the ratio in (A.3):

$$\begin{aligned} &\left(t - \hat{t}\right)s_B s_S + \left(s_B - s_S\right)k \bullet s_G - \left(s_B \hat{w} \bullet s_G - s_S w \bullet s_G\right) \\ &\geq \left(t - \hat{t}\right)s_B s_S + \left(s_B - s_S\right)\left(k \bullet s_G - \hat{w} \bullet s_G\right) \end{aligned} \tag{A.4}$$

The rest of the proof establishes that the right-hand side of inequality (A.4) is non-negative.

First note that, the probability that *B* participates in the set of tail events $T\{B,G\}$ is at least as high as the probability that *S* participates in $T\{S,G\}$. The proof of this inequality is identical to a reasoning behind Theorem 1: since banks *S* and *B* have identical risk characteristics but $s_S < s_B$, *B* is at least as likely to participate in tail events as *S*. This establishes $t \geq \hat{t}$.

It remains to prove that $k \bullet s_G \geq \hat{w} \bullet s_G$ – i.e. that the ES of the set of banks G is at least as large as the portion of the expected losses in $T(\{S,G\})$ associated with banks in G. This involves two steps. First, the probability that any loss configuration associated with *G* (be it in the tail or not) is the same as the probability of that configuration in {S,G}. This reflects the fact that the probability of any loss configuration is independent of the banks that are not in this configuration. Second, an argument similar to that underpinning Fact 1 establishes that: (i) for each tail event that is in *T({S,G}*) and involves banks in *G* (and thus enters the calculation of $\hat{w}$), there is a corresponding tail event that is in *T*(*G*) and features the same banks from *G* (entering the calculation of k); and (ii) the opposite need not be true. ■

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