Modelling volatility interactions in multivariate GARCH models with multiplicative decomposition*

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Abstract

In this paper, I propose a new multivariate GARCH model that brings together two important features of volatilities, non-stationarity and interactions. Assuming a multiplicative decomposition for the variance equations, the new model is constructed to accommodate volatility interactions in the short-term volatilities after accounting for structural changes in the long-term volatilities. The short-term components of the variances are thus described by weakly stationary GARCH processes augmented with cross-market conditional heteroskedasticity, or CMCH, effects. The deterministic functions of time introduce non-stationarity and describe volatility in the long-run. To test for the presence of CMCH effects, I propose a Lagrange Multiplier type test which is reasonably well-behaved in finite samples. Alternative structures are allowed for the correlation component of the new model. It is shown that volatilities of major exchange rate returns tend to be higher and the volatility interactions to be stronger in crisis periods. Volatility persistence decreases by modeling not only the non-stationary nature of volatilities but also their interactions.

Keywords: Volatility interactions; Multivariate GARCH; Long- and short-term variance; Time-varying parameter model; Lagrange multiplier test.

JEL classification: C12, C13, C32, C51.

1 Introduction

Recent multivariate GARCH models are mainly focused on either the non-stationary nature of volatilities or the interactions between volatility processes. In this paper, a new conditional correlation GARCH model is proposed to bring together these two important features of volatilities.

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Models of autoregressive conditional heteroskedasticity (ARCH) and generalized autoregressive conditional heteroskedasticity (GARCH) have been intensively and widely applied to the modelling and forecasting of volatility of daily and weekly financial returns. Among other features, volatility is time-varying and appears to respond to shocks at different frequencies and to be nonstationary. Changes in the volatility process occur not only conditionally but also at different frequencies over time. Even though higher-order GARCH models can be useful for modelling long time series, by allowing the decay of information at different rates, it has been shown that standard GARCH models may not be appropriate to describe nonstationarity when a long span of data is used. This is the case when in the presence of structural changes in the unconditional variance process.

A recent literature on conditional correlation GARCH models is devoted to the decomposition of variances and correlations. The new model introduced in this paper relates to GARCH models with multiplicative decomposition to capture both short- and long-term volatility features. For a recent survey on the multiplicative decomposition of conditional variances and correlations, I refer to Amado *et al.* (2018). Previous multivariate GARCH models with nonstationary GARCH equations were suggested by Amado & Teräsvirta (2014a)[1]. In these models, each variance process is modelled as the multiplicative time-varying (MTV-)GARCH model introduced by Amado & Teräsvirta (2013); see also Amado & Teräsvirta (2008). The short-term subcomponent is assumed to be stationary and is obtained by rescaling the innovations by the long-term subcomponent. This slow moving subcomponent is nonstationary and described by a linear combination of deterministic logistic transition functions of time; see also Amado & Teräsvirta (2013, 2017). The extensions to the multivariate case belong to the general class of multiplicative time-varying conditional correlation (MTV-CC-)GARCH models. The variances are modelled as univariate MTV-GARCH models whereas the conditional correlations have a (potentially) time-varying structure. When conditional correlations have the dynamic structure of Engle (2002), the model becomes the MTV-DCC-GARCH model. If the conditional correlation matrix is defined for higher lags of the volatility standardized residuals as in the varying correlation (VC-)GARCH model of Tse & Tsui (2002), the model is called the MTV-VC-GARCH model. Finally, when the conditional correlations are time-invariant, as in the constant conditional correlation (CCC-)GARCH model of Bollerslev (1990), the model is the MTV-CCC-GARCH model; see Amado & Teräsvirta (2014a) for more details. Another parametric correlation structure, where smooth transitions describe the dynamics of both variances and correlations, was recently introduced by Silvennoinen & Teräsvirta (2017). In the multiplicative time-varying correlation (MTV-TVC-)GARCH model, the unconditional correlations are also modelled as deterministic functions of time. In this model, conditional correlations share the same structure as in the smooth transition conditional correlation (STCC-)GARCH model of Silvennoinen & Teräsvirta (2005, 2015) when time is selected as the transition variable.

[1]Variance decompositions into stationary and nonstationary components were also suggested by, among others, van Bellegem & von Sachs (2004) and Engle & Rangel (2008).

Other generalisations of univariate multiplicative GARCH models to the multivariate case are suggested by Rangel & Engle (2012) for the spline GARCH model of Engle & Rangel (2008) where the dynamics of volatilities are described by both high and low frequency economic variables. Colacito *et al.* (2011) applied the GARCH-MIDAS approach introduced by Engle *et al.* (2013), where the short-term GARCH subcomponent is assumed to move around a long-term subcomponent driven by low frequency realized volatilities.

In the aforementioned multivariate GARCH models, no interactions are assumed in the conditional variances. The volatility component of conditional correlation GARCH models is frequently described by a standard vector GARCH process where the ARCH and GARCH matrices are diagonal. To accommodate volatility spillovers, in the so-called extended (E)CCC-GARCH model introduced by Jeantheau (1998) and its moment structure studied by He & Teräsvirta (2004), the off-diagonal elements of the ARCH and GARCH matrices are allowed to be non-negative[2]. The literature on multivariate GARCH models with multiplicative decomposition accounting for volatility interactions is still scarce. Among the few conditional correlation GARCH models augmented with volatility spillovers, stationarity is usually imposed to the variance equations.

In this paper, I propose a new conditional correlation model with nonstationary GARCH equations and volatility interactions built to capture both short- and long-term dynamics. The model accommodates short-term interactions in volatilities and may be useful to give insight on how they might change during periods of market distress. Each univariate variance process is assumed to have the multiplicative decomposition introduced by Amado & Teräsvirta (2013). The conditional variance subcomponent describes the heteroskedastic behaviour of the return series in the short-run. The positive-valued and deterministic function of time introduces nonstationarity in the volatility process. This is done by allowing the parameters to change smoothly between two extreme volatility states according to a logistic transition function. Stationarity of the conditional variance subcomponent is then obtained by rescaling the innovation series by the long term subcomponent which describes the time-varying baseline volatility.

In the presence of interactions between volatilities, the MTV-GARCH model is misspecified. In particular, I am interested in testing if whether the conditional variance subcomponent should accommodate cross-market interactions in volatilities. As a misspecification test of the MTV-GARCH model, I propose a Lagrange Multiplier (LM-)type test for the presence of short-term cross-market conditional heteroskedastic (CMCH) effects. The resulting model with volatility interactions and general correlation structure shall be called the MTV-ECC-GARCH model, where the "E" stands for extended as in the original ECCC-GARCH model. A Monte Carlo experiment shows that the test performs reasonably well in small samples.

The MTV-ECC-GARCH model is applied to four major exchange rates: the United States dollar, the Australian dollar, the Japanese yen and the British pound against the euro over the 2000's. Studying the daily variation of exchange rates is important as it can be regarded as an

[2] Conrad & Karanasos (2010) show that this restriction can be relaxed and still ensure positive definiteness of the conditional covariance matrix.

uncertainty measure for many macroeconomic variables at the international level, such as relative prices and the value of debt payments in foreign currencies. These affect other closely monitored domestic variables, such as output and unemployment. The presence of volatility interactions in exchange rates has also implications to the conduction of monetary policy, international trade, risk management and portfolio diversification. I find strong evidence for smooth changes in the baseline volatility processes of the Australian dollar, Japanese yen and British pound return series. The baseline volatility decreases over time but eventually reverts towards the higher values observed initially. The LM-type test of volatility interactions shows that all the remaining return series help explain the short-term movements in the volatility of the British pound and so they should be included as covariates in its conditional variance subcomponent. The null hypothesis of a correctly specified MTV-GARCH model is also rejected for the volatility of the Australian dollar, where evidence for short-term interactions with the Japanese yen returns is also found. The different time-varying correlation structures show a common result: a decreasing trend in the comovements between the exchange rate returns.

This paper is organized as follows. In Section 2, I present the new conditional correlation GARCH model with a detailed description of the variance and correlation components as well as the modelling strategy when using nonstationary GARCH equations in multivariate models. Section 3 is devoted to the testing procedure where the LM-type test of volatility interactions is derived and its finite sample properties are studied. The empirical application to major exchange rates is provided in Section 4. Finally, Section 5 concludes this paper.

2 The model

In this section I define the new multivariate conditional correlation GARCH model with nonstationary GARCH equations and volatility interactions. As usually in this class of models, conditional covariances are multiplicatively decomposed into two components, a volatility component and a correlation component. Each variance process is further multiplicatively decomposed into two subcomponents, a stationary subcomponent describing the correlations of volatilities of returns and a nonstationary subcomponent. The correlation component is described by parametric extensions of the constant conditional correlation matrix to account for the dynamics also in the correlations of returns.

Consider the $(m \times 1)$ vector of returns $\mathbf{y}_t$ given by

$$\mathbf{y}_t = \mathsf{E}(\mathbf{y}_t|\mathcal{F}_{t-1}) + \mathbf{u}_t \tag{1}$$

where $\mathbf{u}_t = (u_{1t}, \dots, u_{mt})'$ denotes the vector of innovations and $\mathcal{F}_{t-1}$ contains the information available until time $t-1$. Without loss of generality, I assume $\mathsf{E}(\mathbf{u}_t|\mathcal{F}_{t-1}) = \mathbf{0}_m$ where $\mathbf{0}_m$ denotes an $(m \times 1)$ zero vector, and focus on the conditional covariance matrix of $\mathbf{u}_t$, that is $\mathsf{E}(\mathbf{u}_t\mathbf{u}_t'|\mathcal{F}_{t-1}) = \mathbf{H}_t$. The conditional covariance matrix $\mathbf{H}_t$ has a multiplicative decomposition as follows:

$$\mathbf{H}_t = \mathbf{D}_t \mathbf{S}_t \mathbf{P}_t \mathbf{S}_t \mathbf{D}_t \tag{2}$$

where $\mathbf{D}_t = \mathsf{diag}(\sqrt{h^*_{1t}}, \ldots, \sqrt{h^*_{mt}})$ is a diagonal matrix of conditional standard deviations, $\mathbf{S}_t = \mathsf{diag}(\sqrt{g_{1t}(t/T)}, \ldots, \sqrt{g_{mt}(t/T)})$ is a diagonal matrix of positive-valued and deterministic functions of time where T is the total number of observations. For simplicity, $g_{it}(t/T) \equiv g_{it}$, $i = 1, \ldots, m$. $\mathbf{P}_t$ is a positive definite matrix that contains the time-varying conditional correlations $\rho_{ijt}, i, j = 1, \ldots, m$, with $\rho_{ijt} = 1$ when $i = j$. Furthermore, I assume that $\mathbf{H}_t$ exists and is positive definite. It follows that the innovation process $\mathbf{u}_t$ can be defined as

$$\mathbf{u}_t = \mathbf{D}_t \mathbf{S}_t \mathbf{z}_t \tag{3}$$

where $\mathbf{z}_t = (z_{1t}, \ldots, z_{mt})'$ is a sequence of independent and identically distributed random variables $z_{it} = u_{it}/\sqrt{h^*_{it} g_{it}}$, $i = 1, \ldots, m$. The conditional correlations are thus functions of $\mathbf{z}_t = \mathbf{D}_t^{-1} \boldsymbol{\phi}_t$, where $\boldsymbol{\phi}_t = \mathbf{S}_t^{-1} \mathbf{u}_t$, satisfying $\mathsf{E}(\mathbf{z}_t | \mathcal{F}_{t-1}) = \mathbf{0}_m$ and $\mathsf{E}(\mathbf{z}_t \mathbf{z}_t' | \mathcal{F}_{t-1}) = \mathbf{P}_t$.

2.1 Modelling variances

The volatility component of $\mathbf{H}_t$ is defined as a nonstationary vector GARCH process augmented with CMCH effects. The multivariate GARCH model can thus accommodate interactions in the conditional volatilities by allowing the non-diagonal elements of the ARCH matrix to be positive. This framework follows Francq & Zakoïan (2016) in the sense that each univariate variance process may depend on the past values of all elements in the multivariate series. The model introduced in this paper can be regarded as an extension of their multivariate GARCH model by introducing nonstationarity in the variance equations. This is done by further multiplicatively decomposing the variance processes where smoothly changing deterministic functions of time are introduced.

I start with the multivariate form of the volatility component and then I reduce it to its univariate representation. Set $\mathbf{h}^*_t = (h^*_{1t}, \ldots, h^*_{mt})'$ and $\mathbf{g}_t = (g_{1t}, \ldots, g_{mt})'$. The $(m \times 1)$ vector of variances $\boldsymbol{\sigma}^2_t = (\sigma^2_{1t}, \ldots, \sigma^2_{mt})'$ and has the multiplicative form

$$\boldsymbol{\sigma}^2_t = \mathbf{h}^*_t \odot \mathbf{g}_t \tag{4}$$

where

$$\mathbf{h}^*_t = \boldsymbol{\omega} + \sum_{i=1}^{q} \mathbf{A}_i \boldsymbol{\phi}^2_{t-i} + \sum_{j=1}^{p} \mathbf{B}_j \mathbf{h}^*_{t-j} \tag{5}$$

is the vector of conditional standard deviations h^*_{it}, $i = 1, \ldots, m$,

$$\mathbf{g}_t = \boldsymbol{\delta}_0 + \boldsymbol{\delta}_1 \odot \mathbf{G}(t/T) \tag{6}$$

contains the deterministic and positive-valued functions of time g_{it}, $i = 1, \ldots, m$, and $\odot$ denotes the Hadamard product.

The subcomponent $\mathbf{h}_t^*$ describes conditional heteroskedasticity and the short-term volatility interactions. The elements in this stochastic subcomponent are assumed to follow a weakly stationary GARCH(p, q) process. To accommodate volatility interactions, I assume $\mathbf{A}_i, i = 1, \ldots, q$, is an $(m \times m)$ non-diagonal matrix which contains the ARCH coefficients. Matrix $\mathbf{B}_j, j = 1, \ldots, p$, is a diagonal matrix of m GARCH coefficients. This assumption allows us to have the equation by equation representation of the model. Finally, $\boldsymbol{\omega}$ is an $(m \times 1)$ vector of intercepts. In the nonstationary subcomponent $\mathbf{g}_t$, $\boldsymbol{\delta}_0$ is an $(m \times 1)$ vector of intercepts, $\boldsymbol{\delta}_1$ is an $(m \times 1)$ vector of slope parameters and $\mathbf{G}(t/T) = (G_{1t}(\gamma_1, \boldsymbol{c}_1; t/T), \ldots, G_{mt}(\gamma_m, \boldsymbol{c}_m; t/T))'$ where G_{it} denotes a transition function defined below. The long-term subcomponent g_{it}, the ith element of $\mathbf{g}_t$, introduces nonstationarity and captures the structural changes in the volatility process.

The modelling strategy addressed to $\boldsymbol{\sigma}_t^2$ can always be reduced to the univariate modelling of its elements σ_{it}^2, $i = 1, \ldots, m$. Each variance process $\sigma_{it}^2 = h_{it}^* g_{it}$, similar to the decomposition in the MTV-GARCH model introduced by Amado & Teräsvirta (2013), augmented with the volatility interactions. In the model introduced in this paper, the short-term subcomponent is built to accommodate volatility interactions and is given by

$$h_{it}^* = \omega_i + \sum_{k=1}^{q_i} \sum_{j=1}^{m} \alpha_{kij} \phi_{j,t-k}^2 + \sum_{k=1}^{p_i} \beta_{ki} h_{i,t-k}^*, \quad \omega_i > 0, \alpha_{kij} \geq 0, \beta_{ki} \geq 0, \tag{7}$$

where $\phi_{jt} = u_{jt}/\sqrt{g_{jt}}$, $j = 1, \ldots, m$. This conditional variance subcomponent is assumed to be weakly stationary by rescaling the innovations by the long-term subcomponent g_{it} defined as follows:

$$g_{it} = \delta_{0i} + \delta_{1i} G_{it}(\gamma_i, \mathbf{c}_i; t/T), \quad \delta_{0i} > 0, \delta_{0i} + \delta_{1i} > 0, \tag{8}$$

which is by construction nonstationary and deterministic. Parameter restrictions in (8) are imposed in order to guarantee that g_{it} is positive at every t. Notice that letting δ_{0i} as a free parameter, creates an identification problem. In the multiplicative decomposition of the variance, only one free intercept is allowed. As the solution to this problem, either ω_i or δ_{0i} is fixed to a known positive value in the estimation. The transition function G_{it} is the logistic function with general form

$$G_{it}(\gamma_i, \mathbf{c}_i; s_t) = \left(1 + \exp\left\{-\gamma_i \prod_{k=1}^{r_i} (s_t - c_{ik})\right\}\right)^{-1}, \gamma_i > 0, c_{i1} \leqslant c_{i2} \leqslant \cdots \leqslant c_{ir_i}. \tag{9}$$

where $\mathbf{c}_i$ contains r_i locations of transition, γ_i is the speed of transition and s_t is the transition variable. The dimension of $\mathbf{c}_i$ and γ_i determine, respectively, the shape and smoothness of the logistic transition function. $G_{it}(\gamma_i, \mathbf{c}_i; s_t)$ is continuous for $\gamma_i < \infty$ and bounded between zero and unity. When $\gamma_i \to \infty$, structural breaks can be identified at $c_{i1}, \ldots, c_{ir_i}$. The long-term subcomponent g_{it} is allowed to change smoothly between δ_{0i} in one extreme state and $\delta_{0i} + \delta_{1i}$ in the other extreme. As the decision rule for determining r_i, I test a sequence of nested hypotheses as described in section 3; see Amado & Teräsvirta (2017), Lin & Teräsvirta (1994) and Teräsvirta

(1994). Visual inspection of the series or information criteria can also be used to support the choice of r_i. In this paper, I select time as the transition variable scaled between zero and one, i.e., $s_t \equiv t/T$, where T is the total number of observations, and $G_{it}(\gamma_i, \mathbf{c}_i; t/T) \equiv G_{it}$ for simplicity. Note, however, that any observable variable could be used to govern the transition between the two extreme volatility states. So far, I am considering a single transition function in (8) for modelling the dynamics of the baseline volatility. A linear combination of transition functions may also be used when necessary. A statistical test shall be carried out to check the need of additional transition functions as suggested by Amado & Teräsvirta (2017) for the MTV-GARCH model. This misspecification test of the g_{it} subcomponent and its asymptotic properties are beyond the scope of this paper and only a small empirical exercise is provided in section 4.

2.2 Modelling correlations

Alternative definitions for the correlation component are now discussed. In the GARCH-type correlation structure suggested by Engle (2002), conditional correlations are computed as a function of the past series information and given by the dynamic matrix process

$$\mathbf{Q}_t = (1 - \theta_1 - \theta_2)\overline{\mathbf{Q}} + \theta_1 \mathbf{z}_{t-1}\mathbf{z}'_{t-1} + \theta_2 \mathbf{Q}_{t-1} \tag{10}$$

where $\theta_1 > 0$, $\theta_2 > 0$, $\theta_1 + \theta_2 < 1$ and $\overline{\mathbf{Q}}$ is the unconditional correlation matrix of the volatility standardized residuals $\mathbf{z}_t$. The parameter restrictions are imposed to ensure positive definiteness of $\mathbf{Q}_t$. To obtain valid correlation coefficients, the correlation matrix is rescaled as follows:

$$\mathbf{P}_t = \mathsf{diag}(\mathbf{Q}_t)^{-1/2}\mathbf{Q}_t\mathsf{diag}(\mathbf{Q}_t)^{-1/2}. \tag{11}$$

The model defined in (1)−(11) shall be called the multiplicative time-varying extended dynamic conditional correlation (MTV-EDCC-)GARCH(p, q) model. A special case is nested in this model. When $\mathbf{A}_i$, $i = 1, \ldots, q$, is diagonal and $\mathbf{S}_t \equiv \mathbf{I}$, the identity matrix of order m, the model corresponds to the original DCC-GARCH(p, q) model. For simplicity, $\mathbf{S}(t/T) \equiv \mathbf{S}_t$.

When the interest lies in modelling the long-term or unconditional correlations, I assume the structure introduced by Silvennoinen & Teräsvirta (2005, 2015) for the conditional correlation matrix and use time as the transition variable. In this setting, the deterministic correlations are allowed to vary smoothly between two extreme states as a function of time. Specifically, the correlation matrix $\mathbf{P}_t \equiv \mathbf{P}(t/T)$ and has the form

$$\mathbf{P}(t/T) = \{1 - G_t(\gamma, \mathbf{c}; t/T)\}\,\mathbf{P}_1 + G_t(\gamma, \mathbf{c}; t/T)\mathbf{P}_2 \tag{12}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are positive definite correlation matrices that describe the two extreme correlation states. For each point in time t, the unconditional correlations are computed as an average between these two constant conditional correlations matrices weighted by the logistic transition function defined in (9) (omitting subscript i). The model with the correlation structure (12) is called

Table 1: Summary of multivariate GARCH models nested in the general MTV-ECC-GARCH model.

no volatility interactions & stationary GARCH equations	
CCC-GARCH	Bollerslev (1990)
DCC-GARCH	Engle (2002)
STCC-GARCH	Silvennoinen & Teräsvirta (2005, 2015)
volatility interactions & stationary GARCH equations	
ECCC-GARCH	Jeantheau (1998), He & Teräsvirta (2004) Francq & Zakoïan (2016)
no volatility interactions & nonstationary GARCH equations	
MTV-TVC-GARCH	Silvennoinen & Teräsvirta (2017)
volatility interactions & nonstationary GARCH equations	
MTV-EDCC-GARCH MTV-ETVC-GARCH MTV-ECCC-GARCH	(*this paper*)

the multiplicative time-varying extended time-varying correlation (MTV-ETVC-)GARCH(p, q) model. When $\mathbf{A}_i$, $i = 1, \ldots, q$, is diagonal, the model collapses into the MTV-TVC-GARCH(p, q) model of Silvennoinen & Teräsvirta (2017)[3]. If, additionally, $\mathbf{S}_t \equiv \mathbf{I}$, the model becomes the STCC-GARCH(p, q) model of Silvennoinen & Teräsvirta (2005, 2015), when time is used as the transition variable.

In the simplest model, conditional correlations are assumed to be time-invariant, i.e., $\mathbf{P}_t \equiv \mathbf{P}$ which implies $\mathbf{H}_t = \mathbf{D}_t\mathbf{S}(t/T)\mathbf{P}\mathbf{S}(t/T)\mathbf{D}_t$. The model with the constant correlation structure is called the multiplicative time-varying extended constant conditional correlation (TV-ECCC-)GARCH(p, q) model. When $\mathbf{S}_t \equiv \mathbf{I}$, the model collapses into the original ECCC-GARCH(p, q) model of Jeantheau (1998) and He & Teräsvirta (2004) (assuming $\mathbf{B}_j$, $j = 1, \ldots, p$, is diagonal). In this model, both long-term volatilities and conditional correlations are assumed as time-invariant. If, additionally, $\mathbf{A}_i$, $i = 1, \ldots, q$, is diagonal, the model becomes the CCC-GARCH(p, q) model of Bollerslev (1990). The constant and dynamic extended conditional correlation GARCH models introduced by Francq & Zakoïan (2016) are also nested in, respectively, the MTV-ECCC-GARCH and MTV-EDCC-GARCH models when $\mathbf{S}_t \equiv \mathbf{I}$. These conditional correlation GARCH models are summarized in Table 1.

[3]The first empirical application of the MTV-TVC-GARCH(p, q) model to sovereign bond yields during the most recent European sovereign debt crisis using not only time but also observable financial indicators is provided by Martins & Amado (2018).

2.3 Model selection and estimation strategy

For modelling multivariate series and the comovements of returns and volatilities of returns, I consider the extended conditional correlation GARCH model with multiplicative decomposition described in the previous section. Assuming the conditional mean is known, I focus on the estimation of the volatility and correlation components as in the two-step estimator suggested by Engle (2002). The procedure of testing and modelling the volatility component $\boldsymbol{\sigma}_t^2$ follows the equation by equation strategy suggested by Francq & Zakoïan (2016). Estimation of the correlation component is carried out afterwards conditionally on the estimated volatility processes.

I begin by introducing some notation. Let the set of all parameters be denoted by $\boldsymbol{\theta} = (\boldsymbol{\vartheta}_1', \ldots, \boldsymbol{\vartheta}_m', \boldsymbol{\rho}')'$ where $\boldsymbol{\vartheta}_i$ is the ith set of volatility parameters, $i = 1, \ldots, m$, and $\boldsymbol{\rho}$ contains the correlation parameters. The composition of $\boldsymbol{\rho}$ depends on the correlation structure assumed. Moreover, set $\boldsymbol{\vartheta}_i = (\boldsymbol{\vartheta}_{hi}', \boldsymbol{\vartheta}_{\boldsymbol{g}}', \boldsymbol{\vartheta}_{fi}')'$ where $\boldsymbol{\vartheta}_{hi}$ is the set of standard GARCH parameters in h_{it}^*, $\boldsymbol{\vartheta}_{\boldsymbol{g}} = (\boldsymbol{\vartheta}_{g1}', \ldots, \boldsymbol{\vartheta}_{gm}')'$ where $\boldsymbol{\vartheta}_{gi}$ is the set of parameters in g_{it}, and $\boldsymbol{\vartheta}_{fi}$ contains the CMCH parameters in h_{it}^*. Assuming $\mathbf{u}_t|\mathcal{F}_{t-1} \sim N(\mathbf{0}, \mathbf{H}_t)$, the full conditional log-likelihood function for observation t is defined as

$$\begin{aligned}
\ell_t(\boldsymbol{\theta}, \mathbf{u}_t) &= -\frac{m}{2} \ln 2\pi - \frac{1}{2}\left(\ln|\mathbf{H}_t| + \mathbf{u}_t'\mathbf{H}_t^{-1}\mathbf{u}_t\right) \\
&= -\frac{m}{2} \ln 2\pi - \frac{1}{2}\left(\ln|\mathbf{D}_t\mathbf{S}_t\mathbf{P}_t\mathbf{S}_t\mathbf{D}_t| + \mathbf{u}_t'\mathbf{D}_t^{-1}\mathbf{S}_t^{-1}\mathbf{P}_t^{-1}\mathbf{S}_t^{-1}\mathbf{D}_t^{-1}\mathbf{u}_t\right) \\
&= -\frac{m}{2} \ln 2\pi - \frac{1}{2}\left(2\ln|\mathbf{S}_t| + 2\ln|\mathbf{D}_t| + \ln|\mathbf{P}_t| + \mathbf{z}_t'\mathbf{P}_t^{-1}\mathbf{z}_t\right) \qquad (13)
\end{aligned}$$

where $\mathbf{S}(t/T) \equiv \mathbf{S}_t$ for simplicity.

Equation by equation estimation of the volatility component. Estimate $\boldsymbol{\sigma}_t^2 = (\sigma_{1t}^2(\boldsymbol{\vartheta}_1), \ldots, \sigma_{mt}^2(\boldsymbol{\vartheta}_m))'$ equation by equation where

$$\sigma_{it}^2(\boldsymbol{\vartheta}_i) = h_{it}^*(\boldsymbol{\vartheta}_{hi}, \boldsymbol{\vartheta}_{\boldsymbol{g}}, \boldsymbol{\vartheta}_{fi}) g_{it}(\boldsymbol{\vartheta}_{gi}), \qquad (14)$$

$i = 1, \ldots, m$. Obtain $\hat{\boldsymbol{\vartheta}}_i$, the estimator of $\boldsymbol{\vartheta}_i$, as the solution of

$$\arg\max\; L_{iT}^V(\boldsymbol{\vartheta}_i) = \sum_{t=1}^{T} \ell_{it}(\boldsymbol{\vartheta}_i, \mathbf{u}_t)$$

where

$$\ell_{it}(\boldsymbol{\vartheta}_i, \mathbf{u}_t) = -\frac{1}{2}\left\{\ln(2\pi) + \ln h_{it}^*(\boldsymbol{\vartheta}_{hi}, \boldsymbol{\vartheta}_{\boldsymbol{g}}, \boldsymbol{\vartheta}_{fi}) + \ln g_{it}(\boldsymbol{\vartheta}_{gi}) + \frac{u_{it}^2}{h_{it}^*(\boldsymbol{\vartheta}_{hi}, \boldsymbol{\vartheta}_{\boldsymbol{g}}, \boldsymbol{\vartheta}_{fi}) g_{it}(\boldsymbol{\vartheta}_{gi})}\right\} \qquad (15)$$

is the log-likelihood function of u_{it}, assuming the multiplicative variance decomposition and volatility interactions.

To correctly specify the volatility model that best fits each series within the multivariate system of time series, I propose the following testing and estimation strategy:

1. Test for the presence of conditional heteroskedasticity in the residual series. When residuals are heteroskedastic, it means that the squared residuals are correlated and the volatility model should accommodate this feature. The correct order of the ARCH component is selected by fitting standard GARCH models to the residuals with increasing order until no remaining ARCH effects in the volatility standardized residuals is observed.

2. Assuming no volatility interactions at this stage, test the null hypothesis of constant unconditional variance, i.e., test $\mathsf{H}_{0i}^{AT} : g_{it} \equiv 1$ against the alternative of a smoothly changing long-term volatility structure by means of the test proposed by Amado & Teräsvirta (2014b); see also the brief description of this test in section 3. Consider the two possible outcomes:

   (a) *Failure to reject the null hypothesis of constant unconditional variance.* Set $g_{it} \equiv 1$ in (14) and (15). Check for the presence of volatility interactions by testing the null hypothesis that the ARCH matrix is diagonal; see Pedersen (2017) and Nakatani & Teräsvirta (2010) for testing the diagonality of both ARCH and GARCH matrices. If the null hypothesis is rejected, set $h_{it}^* \equiv h_{it}^*(\boldsymbol{\vartheta}_{hi}, \boldsymbol{\vartheta}_{fi})$ and estimate the GARCH model augmented with the CMCH effects. Otherwise, estimate a standard GARCH model where $h_{it} \equiv h_{it}(\boldsymbol{\vartheta}_{hi})$. The modelling of the univariate volatility process ends at this stage.

   (b) *The null hypothesis of constant unconditional variance is rejected.* The standard GARCH model is misspecified and the structure of g_{it} should be determined. Test a sequence of nested hypotheses as suggested by Amado & Teräsvirta (2014b) to determine the number of location parameters in (9). Estimate the MTV-GARCH model using maximization by parts; see Amado & Teräsvirta (2013) for further details on the estimation. Proceed to the next step.

3. Check the presence of short-term volatility interactions, i.e., test $\mathsf{H}_{0i} : \alpha_{kij} = 0$, $k = 1, \ldots, q_i$, $j = 1, \ldots, m$ and $i \neq j$. Carry out the additive misspecification test of h_{it}, the conditional variance subcomponent in the original MTV-GARCH model, using auxiliary regressions; a full description of this test is provided in section 3. Consider the two possible outcomes:

   (a) *Failure to reject the null hypothesis of no volatility interactions.* The MTV-GARCH model estimated in step 2b is well specified.

   (b) *The null hypothesis of no volatility interactions is rejected.* The MTV-GARCH model is misspecified. Estimate the model augmented with the CMCH effects, with the same number of location parameters obtained from step 2b and using maximization by parts; this estimation procedure is fully described below.

If there is interest in checking the need of additional transition functions, the next step should be added to the modelling procedure.

4. Test the necessity of an additional transition function. In particular, test the null hypothesis $\mathsf{H}_{0i}^2 : g_{it} = \delta_{0i} + \delta_{1i} G_{1it}(\gamma_{1i}, \mathbf{c}_{1i}; t/T)$ against the alternative $\mathsf{H}_{1i}^2 : g_{it} = \delta_{0i} + \sum_{l=1}^{2} \delta_{li} G_{lit}(\gamma_{li}, \mathbf{c}_{li}; t/T)$; see Amado & Teräsvirta (2017) for further details on this type of misspecification for the MTV-GARCH model. Reduce the significance level to half the level used in step 2 and consider the two possible outcomes:

   (a) *Failure to reject the null hypothesis of a single transition.* The time-varying volatility model estimated in the previous step is well specified.

   (b) *The null hypothesis of a single transition is rejected.* Choose the type of the second transition, i.e., $r_i = 1, 2$ or 3 in G_{2it}. Re-estimate the time-varying volatility model with two transition functions using the procedure described below.

   Repeat this step until no additional transition functions are necessary. The modelling of the univariate volatility process ends at this stage.

Repeat the modelling strategy 1–3 (or 1–4) for each series of residuals and compute the vector of volatility standardised residuals $\hat{\mathbf{z}}_t = (\hat{z}_{1t}, \ldots, \hat{z}_{mt})'$ where $\hat{z}_{it} = u_{it}/\sqrt{(\hat{h}_{it}^* \hat{g}_{it})}$.

Estimation of the correlation component. Obtain $\hat{\boldsymbol{\rho}}$ by quasi maximum likelihood as the solution of

$$\arg\max\, L_T^C(\boldsymbol{\rho}|\hat{\boldsymbol{\vartheta}}_1, \ldots, \hat{\boldsymbol{\vartheta}}_m) = -\frac{1}{2}\sum_{t=1}^{T}\left(\ln|\mathbf{P}_t| + \hat{\mathbf{z}}_t' \mathbf{P}_t^{-1} \hat{\mathbf{z}}_t\right). \tag{16}$$

The two-step estimator just described completes the first iteration in the multi-step estimator. Note that to obtain fully efficient estimations, volatility and correlation parameters should be iteratively estimated until convergence by full maximum likelihood.

For the conditional variance processes with the multiplicative decomposition (14), maximization by parts should be applied as follows:

1. By setting $h_{it}^{*(0)} = 1$, obtain $\hat{\boldsymbol{\vartheta}}_{gi}^{(0)}$ and compute $\hat{\phi}_{it}^{(0)} = u_{it}/\sqrt{\hat{g}_{it}(\hat{\boldsymbol{\vartheta}}_{gi}^{(0)})}$. The parameters γ_i and δ_{0i} are fixed to, respectively, $\hat{\gamma}_i^{(0)}$ and $\hat{\delta}_{0i}^{(0)}$ from this iteration onwards. Repeat this step for $i = 1, \ldots, m$ and obtain $\hat{\boldsymbol{\vartheta}}_{\boldsymbol{g}}^{(0)} = (\hat{\boldsymbol{\vartheta}}_{g1}^{(0)\prime}, \ldots, \hat{\boldsymbol{\vartheta}}_{gm}^{(0)\prime})'$ and $\hat{\boldsymbol{\phi}}_t^{(0)} = (\hat{\phi}_{1t}^{(0)}, \ldots, \hat{\phi}_{mt}^{(0)})'$ necessary to estimate $\boldsymbol{\vartheta}_{hi}$ and $\boldsymbol{\vartheta}_{fi}$.

2. Obtain $\hat{\boldsymbol{\vartheta}}_{hi}^{(1)}$ and $\hat{\boldsymbol{\vartheta}}_{fi}^{(1)}$ and compute $\hat{h}_{it}^{*(1)} = \hat{h}_{it}^*(\hat{\boldsymbol{\vartheta}}_{hi}^{(1)}, \hat{\boldsymbol{\vartheta}}_{\boldsymbol{g}}^{(0)}, \hat{\boldsymbol{\vartheta}}_{fi}^{(1)})$. Repeat this step for $i = 1, \ldots, m$.

3. Re-estimate $\boldsymbol{\vartheta}_{gi}$ and obtain $\hat{\boldsymbol{\vartheta}}_{gi}^{(1)}$. Update $\hat{\boldsymbol{\vartheta}}_{\boldsymbol{g}}$ and $\hat{\boldsymbol{\phi}}_t$ by, respectively, replacing $\hat{\boldsymbol{\vartheta}}_{gi}^{(0)\prime}$ by $\hat{\boldsymbol{\vartheta}}_{gi}^{(1)\prime}$ in $\hat{\boldsymbol{\vartheta}}_{\boldsymbol{g}}^{(0)}$ and $\hat{\phi}_{it}^{(0)}$ by $\hat{\phi}_{it}^{(1)} = u_{it}/\sqrt{\hat{g}_{it}(\hat{\boldsymbol{\vartheta}}_{gi}^{(1)})}$ in $\hat{\boldsymbol{\phi}}_t^{(0)}$. Repeat this procedure for $i = 1, \ldots, m$ to obtain $\hat{\boldsymbol{\vartheta}}_{\boldsymbol{g}}^{(1)} = (\hat{\boldsymbol{\vartheta}}_{g1}^{(1)\prime}, \ldots, \hat{\boldsymbol{\vartheta}}_{gm}^{(1)\prime})'$ and $\hat{\boldsymbol{\phi}}_t^{(1)} = (\hat{\phi}_{1t}^{(1)}, \ldots, \hat{\phi}_{mt}^{(1)})'$.

4. Iterate steps 2 and 3 until convergence[4]. In each iteration $n > 1$, one obtains $\hat{\boldsymbol{\vartheta}}_{hi}^{(n)} =$

[4]Convergence is usually reached after a small number of iterations.

$(\hat{\omega}_i^{(n)}, \hat{\alpha}_i^{(n)}, \hat{\beta}_i^{(n)})'$, $\{\hat{\boldsymbol{\vartheta}}_{fi}^{(n)}\} = \hat{\alpha}_{ij}^{(n)}$, $\hat{\boldsymbol{\vartheta}}_{gi}^{(n)} = (\hat{\delta}_{0i}^{(0)}, \hat{\delta}_{1i}^{(n)}, \hat{\gamma}_i^{(0)}, \hat{\boldsymbol{c}}_i^{(n)\prime})'$, $\hat{h}_{it}^{*(n)} = \hat{h}_{it}^{*}(\hat{\boldsymbol{\vartheta}}_{hi}^{(n)}, \hat{\boldsymbol{\vartheta}}_{\boldsymbol{g}}^{(n-1)}, \hat{\boldsymbol{\vartheta}}_{fi}^{(n)})$, and $\hat{g}_{it}^{(n)} = g_{it}(\hat{\boldsymbol{\vartheta}}_{gi}^{(n)})$, $i, j = 1, \ldots, m$ and $i \neq j$.

Francq & Zakoïan (2016) showed strong consistency and asymptotic normality of the equation by equation estimator for fairly general conditional correlation GARCH models with volatility interactions. Ling & McAleer (2003) also proved the asymptotic properties of the CCC-GARCH model without imposing any diagonality for the ARCH and GARCH matrices. However, in both cases, stationarity is imposed to the conditional variances. For inference, I assume that the asymptotic distribution of the estimator $\hat{\boldsymbol{\vartheta}}_i, i = 1, \ldots, m$, is normal.

3 The test of CMCH effects

When modelling short-term volatility interactions, it is necessary to first check the hypothesis of constant unconditional (long-term) variance. The first test implemented in step 2 of the equation by equation estimation of the volatility component in section 2.3, is the univariate test suggested by Amado & Teräsvirta (2014b) for testing the null hypothesis of constant unconditional variance. When this null hypothesis holds, a standard GARCH model is sufficient to describe the dynamics of volatility. Under the alternative, the time-varying unconditional variance is modelled as a deterministic function of time. Rejecting the null thus provides evidence for nonstationarity in the volatility process meaning that a standard GARCH model is not appropriate to fit the data. For that reason, this test can also be regarded as a general misspecification test of the GARCH model. Before proceeding to the test for short-term volatility interactions, I provide a brief description of this testing procedure. For further details I refer to Amado & Teräsvirta (2014b).

Consider a vector GARCH$(1,1)$ process by setting $p = q = 1$ in (5) with $\mathbf{A}_1$ and $\mathbf{B}_1$ denoting diagonal ARCH and GARCH matrices, respectively. The choice of a first-order GARCH model is reasonable since it is usually sufficient to capture the heteroskedastic behaviour of financial time series. Under the alternative, the variance of u_{it} follows an MTV-GARCH$(1,1)$ model. Hence, $\sigma_{it}^2 = h_{it} g_{it}$ where the conditional variance component

$$h_{it} = \omega_i + \alpha_i \phi_{i,t-1}^2 + \beta_i h_{i,t-1}, \quad \omega_i > 0, \alpha_i \geq 0, \beta_i \geq 0, \tag{17}$$

with $\phi_{it} = u_{it}/\sqrt{g_{it}}$ and the deterministic component g_{it} with a single transition function is defined in (8). For this MTV-GARCH$(1,1)$ process, the subsets of volatility parameters are defined as follows: $\boldsymbol{\vartheta}_{hi} = (\omega_i, \alpha_i, \beta_i)'$ and $\boldsymbol{\vartheta}_{gi} = (\delta_{0i}, \delta_{1i}, \gamma_i, \boldsymbol{c}_i')'$ (note that $\boldsymbol{\vartheta}_{fi} = \mathbf{0}_{m-1}$). When $g_{it} \equiv 1$, $\phi_{it} = u_{it}$ and the model collapses into the standard GARCH model in which the unconditional variance is assumed as constant. Then, to check the constancy of the unconditional variance, I can test either $\mathsf{H}_{0i}^{AT} : g_{it} \equiv 1$ or $\mathsf{H}_{0i}^{AT} : \gamma_i = 0$ against $\mathsf{H}_{1i}^{AT} : g_{it} = 1 + \delta_{1i} G_{it}(\gamma_i, \mathbf{c}_i; t/T)$. A fairly flexible form is assumed for $G_{it}(\gamma_i, \mathbf{c}_i; t/T)$ by setting $r_i = 3$, i.e., $\boldsymbol{c}_i = (c_{1i}, c_{2i}, c_{3i})'$. Due to the inability for identifying δ_{1i} and $\boldsymbol{c}_i$ under H_{0i}^{AT}, the transition function is replaced by its first-order Taylor approximation; see Luukkonen *et al.* (1988). The resulting linearised and re-parametrised deterministic component is given by

$$g_{it}^* = \delta_{0i}^* + (t/T)\delta_{1i}^* + (t/T)^2\delta_{2i}^* + (t/T)^3\delta_{3i}^* \tag{18}$$

where δ_{0i}^*, δ_{1i}^*, δ_{2i}^* and δ_{3i}^* are functions of the original parameters. When H_{0i}^{AT} holds, $\delta_{1i}^* = \delta_{2i}^* = \delta_{3i}^* = 0$ and the parameters in (17), denoted by $\boldsymbol{\vartheta}_{hi}$, remain constant. H_{0i}^{AT} is thus replaced by the auxiliary null hypothesis $\mathsf{H}_{0i}^{AT\prime} : \delta_{1i}^* = \delta_{2i}^* = \delta_{3i}^* = 0$. Rejection of $\mathsf{H}_{0i}^{AT\prime}$ implies testing the following sequence of nested hypotheses to specify the correct number of locations in the transition function:

$$\begin{aligned} &\mathsf{H}_{03i}^{AT} : \delta_{3i}^* = 0 \\ &\mathsf{H}_{02i}^{AT} : \delta_{2i}^* = 0 \quad | \quad \delta_{3i}^* = 0 \\ &\mathsf{H}_{01i}^{AT} : \delta_{1i}^* = 0 \quad | \quad \delta_{3i}^* = \delta_{2i}^* = 0. \end{aligned}$$

The test results can be obtained using auxiliary regressions by the so-called TR^2 test statistics; see Wooldridge (1990, 1991). Under the null hypothesis, these tests have an asymptotic χ^2 distribution with three degrees of freedom in the general misspecification test and one degree of freedom in each of the sequential tests.

In the MTV-GARCH model, cross-market effects across volatility processes are however not allowed and so volatility is explained only by past series information. When volatility interactions occur in multivariate series, a time-varying GARCH model is not sufficient to model the daily variation in the data. Specifically, I am interested in the case where h_{it} is additively misspecified. To test this hypothesis I propose an LM-type test. To derive the test statistics, I start by defining the augmented MTV-GARCH$(1,1)$ model, where CMCH effects are added to h_{it}. In the resulting extended model, the conditional variance subcomponent, denoted as h_{it}^*, has the form

$$h_{it}^* = \omega_i + \sum_{j=1}^{m} \alpha_{ij}\phi_{j,t-1}^2 + \beta_i h_{i,t-1}^*, \quad \omega_i > 0, \alpha_{ij} \geq 0, j = 1, \ldots, m, \beta_i \geq 0. \tag{19}$$

This equation is nested in (7) for $q_i = p_i = 1$ and with subscripts k dropped for simplicity. The subsets of volatility parameters become $\boldsymbol{\vartheta}_{hi} = (\omega_i, \alpha_i, \beta_i)'$, $\boldsymbol{\vartheta}_{\mathbf{g}} = (\boldsymbol{\vartheta}_{g1}, \ldots, \boldsymbol{\vartheta}_{gm})'$ with $\boldsymbol{\vartheta}_{gi} = (\delta_{1i}, \mathbf{c}_i')'$[5], and $\boldsymbol{\vartheta}_{fi} = \{\alpha_{ij}\}$, $j = 1, \ldots, m$ and $i \neq j$. It is useful to define $h_{it}^* \equiv h_{it}^*(\boldsymbol{\vartheta}_{hi}, \boldsymbol{\vartheta}_{\mathbf{g}}, \boldsymbol{\vartheta}_{fi}; \mathbf{u}_t)$ as the unconstrained conditional variance subcomponent (with volatility interactions) and $h_{it} \equiv h_{it}(\boldsymbol{\vartheta}_{hi}, \boldsymbol{\vartheta}_{gi}; u_{it})$ as the constrained conditional variance subcomponent (with no volatility interactions). Finally, $g_{it} \equiv g_{it}(\boldsymbol{\vartheta}_{gi})$ is the deterministic subcomponent common to both unconstrained and constrained models. When $\boldsymbol{\vartheta}_{fi} = \mathbf{0}_{m-1}$, the model becomes the MTV-GARCH model. To check for this additive misspecification of the h_{it} subcomponent, I test the null hypothesis

$$\mathsf{H}_{0i} : \alpha_{ij} = 0, \quad \text{for all } j = 1, \ldots, m \text{ and } i \neq j, \tag{20}$$

[5]Parameters δ_{0i} and γ_i are omitted since they are fixed from the first iteration onwards in the estimation.

against the alternative,

$$\mathsf{H}_{1i}: \text{ at least one } \alpha_{ij} > 0. \tag{21}$$

The LM-type test statistic for testing the null hypothesis is given in the following theorem.

Theorem 3.1. *Consider the multiplicative variance decomposition $\sigma_{it}^2 = h_{it}^* g_{it}$ whose components are defined in (19) and (8). Under $\mathsf{H}_{0i}: \boldsymbol{\vartheta}_{fi} = \mathbf{0}_{m-1}$, the LM-type statistic*

$$T\overline{\mathbf{s}}_i'(\hat{\boldsymbol{\vartheta}}_i)_{[\boldsymbol{\vartheta}_f]}\hat{\mathcal{I}}_i^{-1}(\hat{\boldsymbol{\vartheta}}_i)_{[\boldsymbol{\vartheta}_f,\boldsymbol{\vartheta}_f]}\overline{\mathbf{s}}_i(\hat{\boldsymbol{\vartheta}}_i)_{[\boldsymbol{\vartheta}_f]} \xrightarrow{d} \chi_{m-1}^2 \tag{22}$$

where $\overline{\mathbf{s}}_i'(\hat{\boldsymbol{\vartheta}}_i)_{[\boldsymbol{\vartheta}_f]}$ and $\hat{\mathcal{I}}_i^{-1}(\hat{\boldsymbol{\vartheta}}_i)_{[\boldsymbol{\vartheta}_f,\boldsymbol{\vartheta}_f]}$ is the relevant block of, respectively, the score and the consistently estimated population information matrix evaluated at $\boldsymbol{\vartheta}_i = \hat{\boldsymbol{\vartheta}}_i$, and $m-1$ is the number of degrees of freedom.

Proof. Consider the full volatility parameter vector $\boldsymbol{\vartheta}_i = (\boldsymbol{\vartheta}_{hi}', \boldsymbol{\vartheta}_{\mathbf{g}}', \boldsymbol{\vartheta}_{fi}')'$ and the conditional log-likelihood function defined in (15), assuming a normal distribution for u_{it}. The partitioned score of (15) at each point in time t is given by

$$\mathbf{s}_{it}(\boldsymbol{\vartheta}_i) = (\mathbf{s}_{hit}(\boldsymbol{\vartheta}_i)', \mathbf{s}_{\mathbf{g}t}(\boldsymbol{\vartheta}_i)', \mathbf{s}_{fit}(\boldsymbol{\vartheta}_i)')' \tag{23}$$

where $\mathbf{s}_{hit}(\boldsymbol{\vartheta}_i) = \partial\ell_{it}(\boldsymbol{\vartheta}_i;\mathbf{u}_t)/\partial\boldsymbol{\vartheta}_{hi}$, $\mathbf{s}_{\mathbf{g}t}(\boldsymbol{\vartheta}_i) = \left\{\partial\ell_{it}(\boldsymbol{\vartheta}_i;\mathbf{u}_t)/\partial\boldsymbol{\vartheta}_{g1}', \ldots, \partial\ell_{it}(\boldsymbol{\vartheta}_i;\mathbf{u}_t)/\partial\boldsymbol{\vartheta}_{gm}'\right\}'$ and, lastly, $\mathbf{s}_{fit}(\boldsymbol{\vartheta}_i) = \partial\ell_{it}(\boldsymbol{\vartheta}_i;\mathbf{u}_t)/\partial\boldsymbol{\vartheta}_{fi}$. The analytical expressions for the first-order partial derivatives of $\ell_{it}(\boldsymbol{\vartheta}_i;\mathbf{u}_t)$ with respect to each volatility parameter subset have the form

$$\frac{\partial\ell_{it}(\boldsymbol{\vartheta}_i;\mathbf{u}_t)}{\partial\boldsymbol{\vartheta}_{hi}} = (z_{it}^2-1)\frac{1}{2h_{it}^*}\frac{\partial h_{it}^*}{\partial\boldsymbol{\vartheta}_{hi}} \tag{24}$$

$$\frac{\partial\ell_{it}(\boldsymbol{\vartheta}_i;\mathbf{u}_t)}{\partial\boldsymbol{\vartheta}_{gi}} = (z_{it}^2-1)\left(\frac{1}{2g_{it}}\frac{\partial g_{it}}{\partial\boldsymbol{\vartheta}_{gi}} + \frac{1}{2h_{it}^*}\frac{\partial h_{it}^*}{\partial\boldsymbol{\vartheta}_{gi}}\right) \tag{25}$$

$$\frac{\partial\ell_{it}(\boldsymbol{\vartheta}_i;\mathbf{u}_t)}{\partial\boldsymbol{\vartheta}_{gj}} = (z_{it}^2-1)\left(\frac{1}{2h_{it}^*}\frac{\partial h_{it}^*}{\partial\boldsymbol{\vartheta}_{gj}}\right) \tag{26}$$

$$\frac{\partial\ell_{it}(\boldsymbol{\vartheta}_i;\mathbf{u}_t)}{\partial\boldsymbol{\vartheta}_{fi}} = (z_{it}^2-1)\frac{1}{2h_{it}^*}\frac{\partial h_{it}^*}{\partial\boldsymbol{\vartheta}_{fi}}, \tag{27}$$

$j = 1, \ldots, m$ and $i \neq j$, where $z_{it} = u_{it}/\sqrt{h_{it}^* g_{it}}$. Set $\mathbf{d}_{\mathbf{c}_i} = (d_{c_{i1}}, \ldots, d_{c_{ir_i}})'$ where

$$d_{c_{ik}} = \frac{\partial g_{it}}{\partial c_{ik}} = -\gamma_i\delta_{1i}G_{it}(1-G_{it})\prod_{l=1}^{r_i-1}(s_t - c_{il}),$$

$k, l = 1, \ldots, r_i$ and $l \neq k$. The first-order partial derivatives of h_{it}^* and g_{it} with respect to each volatility parameter subset are given by

$$\frac{\partial h^*_{it}}{\partial \boldsymbol{\vartheta}_{hi}} = \mathbf{v}_{i,t-1} + \beta_i \frac{\partial h^*_{i,t-1}}{\partial \boldsymbol{\vartheta}_{hi}} \tag{28}$$

$$\frac{\partial g_{it}}{\partial \boldsymbol{\vartheta}_{gi}} = (G_{it}, \mathbf{d}'_{\mathbf{c}_i})' \tag{29}$$

$$\frac{\partial h^*_{it}}{\partial \boldsymbol{\vartheta}_{gj}} = -\alpha_{ij}\phi^2_{j,t-1}\frac{1}{g_{j,t-1}}\frac{\partial g_{j,t-1}}{\partial \boldsymbol{\vartheta}_{gj}} \tag{30}$$

$$\frac{\partial h^*_{it}}{\partial \boldsymbol{\vartheta}_{fi}} = \boldsymbol{\phi}^2_{-i,t-1} + \beta_i \frac{\partial h^*_{i,t-1}}{\partial \boldsymbol{\vartheta}_{fi}} \tag{31}$$

where $\mathbf{v}_{it} = (1, \phi^2_{it}, h^*_{it})'$, $G_{it} \equiv G_{it}(\gamma_i, \mathbf{c}_i; t/T)$ and $\boldsymbol{\phi}_{-i,t} = \{\phi_{jt}\}, j = 1, \ldots, m$ and $i \neq j$. In words, if $\boldsymbol{\phi}_t = (\phi_{1t}, \ldots, \phi_{mt})'$, $\boldsymbol{\phi}_{-i,t}$ contains all its elements but the ith element.

When H_{0i} holds, $\boldsymbol{\vartheta}_{fi} = \mathbf{0}_{m-1}$ and $h^*_{it} = h_{it} \equiv h_{it}(\boldsymbol{\vartheta}_{hi}, \boldsymbol{\vartheta}_{gi}; u_{it})$. Set $\hat{h}^0_{it} = h_{it}(\hat{\boldsymbol{\vartheta}}_{hi}, \hat{\boldsymbol{\vartheta}}_{gi}; u_{it})$ as the estimated constrained conditional variance subcomponent that is valid under the null hypothesis, $\hat{g}^0_{it} = g_{it}(\hat{\boldsymbol{\vartheta}}_{gi})$ as the estimated deterministic subcomponent. Denote $\hat{\boldsymbol{\vartheta}}_{hi} = (\hat{\omega}_i, \hat{\alpha}_i, \hat{\beta}_i)'$ and $\hat{\boldsymbol{\vartheta}}_{gi} = (\hat{\delta}_{0i}, \hat{\delta}_{1i}, \hat{\gamma}_i, \hat{\boldsymbol{c}}'_i)'$ as the vectors of estimated volatility parameters in, respectively, the constrained conditional variance equation h_{it} and deterministic function g_{it}. Under regularity conditions, Amado & Teräsvirta (2013) showed that maximization by parts, where the estimated $\hat{\boldsymbol{\vartheta}}_{hi}$ and $\hat{\boldsymbol{\vartheta}}_{gi}$ are obtained iteratively until convergence, leads to consistent and asymptotically normal estimates. If $\boldsymbol{\vartheta}^0_{hi}$ and $\boldsymbol{\vartheta}^0_{gi}$ are, respectively, the vectors of the true values of $\boldsymbol{\vartheta}_{hi}$ and $\boldsymbol{\vartheta}_{gi}$, they showed that $\hat{\boldsymbol{\vartheta}}_{hi} \rightarrow \boldsymbol{\vartheta}^0_{hi}$ and $\hat{\boldsymbol{\vartheta}}_{gi} \rightarrow \boldsymbol{\vartheta}^0_{gi}$ in probability as $T \rightarrow \infty$. It follows that, under H_{0i},

$$\left.\frac{\partial \hat{\ell}_{it}(\hat{\boldsymbol{\vartheta}}_i; \mathbf{u}_t)}{\partial \boldsymbol{\vartheta}_{hi}}\right|_{\mathsf{H}_{0i}} = (\hat{z}^2_{it} - 1)\frac{1}{2\hat{h}_{it}}\left.\frac{\partial \hat{h}^*_{it}}{\partial \boldsymbol{\vartheta}_{hi}}\right|_{\mathsf{H}_{0i}} \tag{32}$$

$$\left.\frac{\partial \hat{\ell}_{it}(\hat{\boldsymbol{\vartheta}}_i; \mathbf{u}_t)}{\partial \boldsymbol{\vartheta}_{gi}}\right|_{\mathsf{H}_{0i}} = (\hat{z}^2_{it} - 1)\left(\frac{1}{2\hat{g}^0_{it}}\left.\frac{\partial \hat{g}_{it}}{\partial \boldsymbol{\vartheta}_{gi}}\right|_{\mathsf{H}_{0i}} + \frac{1}{2\hat{h}^0_{it}}\left.\frac{\partial \hat{h}^*_{it}}{\partial \boldsymbol{\vartheta}_{gi}}\right|_{\mathsf{H}_{0i}}\right) \tag{33}$$

$$\left.\frac{\partial \hat{\ell}_{it}(\hat{\boldsymbol{\vartheta}}_i; \mathbf{u}_t)}{\partial \boldsymbol{\vartheta}_{gj}}\right|_{\mathsf{H}_{0i}} = (\hat{z}^2_{it} - 1)\frac{1}{2\hat{h}^0_{it}}\left.\frac{\partial \hat{h}^*_{it}}{\partial \boldsymbol{\vartheta}_{gj}}\right|_{\mathsf{H}_{0i}} \tag{34}$$

$$\left.\frac{\partial \hat{\ell}_{it}(\hat{\boldsymbol{\vartheta}}_i; \mathbf{u}_t)}{\partial \boldsymbol{\vartheta}_{fi}}\right|_{\mathsf{H}_{0i}} = (\hat{z}^2_{it} - 1)\frac{1}{2\hat{h}^0_{it}}\left.\frac{\partial \hat{h}^*_{it}}{\partial \boldsymbol{\vartheta}_{fi}}\right|_{\mathsf{H}_{0i}} \tag{35}$$

where $\hat{z}_{it} = u_{it}/\sqrt{\hat{h}^0_{it}\hat{g}^0_{it}}$, $j = 1, \ldots, m$ and $i \neq j$. Analogously, the partial derivatives (28)–(29)

simplify to

$$\left.\frac{\partial \hat{h}_{it}^{*}}{\partial \boldsymbol{\vartheta}_{hi}}\right|_{\mathsf{H}_{0i}} = \hat{\mathbf{v}}_{i,t-1} + \hat{\beta}_i \left.\frac{\partial \hat{h}_{i,t-1}^{*}}{\partial \boldsymbol{\vartheta}_{hi}}\right|_{\mathsf{H}_{0i}} \tag{36}$$

$$\left.\frac{\partial \hat{h}_{it}^{*}}{\partial \boldsymbol{\vartheta}_{gi}}\right|_{\mathsf{H}_{0i}} = -\hat{\alpha}_i \hat{\phi}_{i,t-1}^2 \frac{1}{\hat{g}_{i,t-1}^0} \left.\frac{\partial \hat{g}_{i,t-1}}{\partial \boldsymbol{\vartheta}_{gi}}\right|_{\mathsf{H}_{0i}} \tag{37}$$

$$\left.\frac{\partial \hat{h}_{it}^{*}}{\partial \boldsymbol{\vartheta}_{gj}}\right|_{\mathsf{H}_{0i}} = 0$$

$$\left.\frac{\partial \hat{g}_{it}}{\partial \boldsymbol{\vartheta}_{gi}}\right|_{\mathsf{H}_{0i}} = (\hat{G}_{it}, \hat{\mathbf{d}}_{\mathbf{c}_i}')' \tag{38}$$

$$\left.\frac{\partial \hat{h}_{it}^{*}}{\partial \boldsymbol{\vartheta}_{fi}}\right|_{\mathsf{H}_{0i}} = \hat{\boldsymbol{\phi}}_{-i,t-1}^2 + \hat{\beta}_i \left.\frac{\partial \hat{h}_{i,t-1}^{*}}{\partial \boldsymbol{\vartheta}_{fi}}\right|_{\mathsf{H}_{0i}} \tag{39}$$

where $\hat{\mathbf{v}}_{it} = (1, \hat{\phi}_{it}, \hat{h}_{it}^0)'$ with $\hat{\phi}_{it} = u_{it}/\sqrt{\hat{g}_{it}^0}$, $\{\hat{\boldsymbol{\phi}}_{-i,t}\} = \hat{\phi}_{jt}$, $j = 1, \ldots, m$ and $i \neq j$, $\hat{G}_{it} \equiv \hat{G}_{it}(\hat{\gamma}_i, \hat{\mathbf{c}}_i; t/T)$ and, finally,

$$\hat{d}_{c_{ik}} = \frac{\partial \hat{g}_{it}}{\partial c_{ik}} = -\hat{\gamma}_i \hat{\delta}_{1i} \hat{G}_{it}(1 - \hat{G}_{it}) \prod_{l=1}^{r_i - 1} (s_t - \hat{c}_{il}).$$

The population information matrix $\mathcal{I}_i(\boldsymbol{\vartheta}_i^0)$ is given by the expectation of the outer product of the score evaluated at the true volatility parameter vector $\boldsymbol{\vartheta}_i^0 = (\boldsymbol{\vartheta}_{hi}^{0\prime}, \boldsymbol{\vartheta}_{\mathbf{g}}^{0\prime}, \boldsymbol{\vartheta}_{fi}^{0\prime})'$ such that

$$\mathcal{I}_i(\boldsymbol{\vartheta}_i^0) = \mathsf{E}\left\{\mathbf{s}_{it}(\boldsymbol{\vartheta}_i^0)\mathbf{s}_{it}(\boldsymbol{\vartheta}_i^0)'\right\} = \begin{bmatrix} \mathcal{I}_i^{hh}(\boldsymbol{\vartheta}_i^0) & \mathcal{I}_i^{hg}(\boldsymbol{\vartheta}_i^0) & \mathcal{I}_i^{hf}(\boldsymbol{\vartheta}_i^0) \\ \mathcal{I}_i^{gh}(\boldsymbol{\vartheta}_i^0) & \mathcal{I}_i^{gg}(\boldsymbol{\vartheta}_i^0) & \mathcal{I}_i^{gf}(\boldsymbol{\vartheta}_i^0) \\ \mathcal{I}_i^{fh}(\boldsymbol{\vartheta}_i^0) & \mathcal{I}_i^{fg}(\boldsymbol{\vartheta}_i^0) & \mathcal{I}_i^{ff}(\boldsymbol{\vartheta}_i^0) \end{bmatrix}. \tag{40}$$

Ling & McAleer (2003) showed that $\mathcal{I}_i(\boldsymbol{\vartheta}_i^0)$ can be consistently estimated by

$$\hat{\mathcal{I}}_i(\boldsymbol{\vartheta}_i) = T^{-1} \sum_{t=1}^{T} \left\{\mathbf{s}_{it}(\boldsymbol{\vartheta}_i)\mathbf{s}_{it}(\boldsymbol{\vartheta}_i)'\right\} \tag{41}$$

and from where

$$\hat{\mathcal{I}}_i^{-1}(\hat{\boldsymbol{\vartheta}}_i)_{[\boldsymbol{\vartheta}_{fi}, \boldsymbol{\vartheta}_{fi}]} = \left\{ \begin{bmatrix} \hat{\mathcal{I}}_i^{hh}(\hat{\boldsymbol{\vartheta}}_i) & \hat{\mathcal{I}}_i^{hg}(\hat{\boldsymbol{\vartheta}}_i) \\ \hat{\mathcal{I}}_i^{gh}(\hat{\boldsymbol{\vartheta}}_i) & \hat{\mathcal{I}}_i^{gg}(\hat{\boldsymbol{\vartheta}}_i) \end{bmatrix} - \begin{bmatrix} \hat{\mathcal{I}}_i^{hf}(\hat{\boldsymbol{\vartheta}}_i) \\ \hat{\mathcal{I}}_i^{gf}(\hat{\boldsymbol{\vartheta}}_i) \end{bmatrix} \left[\hat{\mathcal{I}}_i^{ff}(\hat{\boldsymbol{\vartheta}}_i)\right]^{-1} \left[\hat{\mathcal{I}}_i^{fh}(\hat{\boldsymbol{\vartheta}}_i) \quad \hat{\mathcal{I}}_i^{fg}(\hat{\boldsymbol{\vartheta}}_i)\right] \right\}^{-1} \tag{42}$$

can be obtained by applying the inverse of a partitioned matrix. By denoting $\bar{\mathbf{s}}_{fi}(\hat{\boldsymbol{\vartheta}}_i) = T^{-1}\sum_{t=1}^{T} \mathbf{s}_{fit}(\hat{\boldsymbol{\vartheta}}_i)$ as the average score for the CMCH parameters $\boldsymbol{\vartheta}_{fi}$ evaluated at the estimated parameter values, I obtain the LM-type statistic (22). ■

To carry out the test using auxiliary regressions, set $\hat{\mathbf{x}}_{it}^{hh} = (\hat{h}_{it}^0)^{-1}\partial\hat{h}_{it}/\partial\boldsymbol{\vartheta}_{hi}|_{\mathsf{H}_{0i}}$, $\hat{\mathbf{x}}_{it}^{hg} = (\hat{h}_{it}^0)^{-1}\partial\hat{h}_{it}/\partial\boldsymbol{\vartheta}_{gi}|_{\mathsf{H}_{0i}}$, $\hat{\mathbf{x}}_{it}^{hf} = (\hat{h}_{it}^0)^{-1}\partial\hat{h}_{it}/\partial\boldsymbol{\vartheta}_{fi}|_{\mathsf{H}_{0i}}$ and $\hat{\mathbf{x}}_{it}^{gg} = (\hat{g}_{it}^0)^{-1}\partial\hat{g}_{it}/\partial\boldsymbol{\vartheta}_{gi}|_{\mathsf{H}_{0i}}$. The so-called non-robust TR^2 version of test statistic (22) can be carried out according to the following steps:

1. Consistently estimate the MTV-GARCH$(1,1)$ model by maximization by parts, save the series of estimated volatility standardized residuals $\hat{z}_{it} = u_{it}/\sqrt{\hat{h}_{it}^0\hat{g}_{it}^0}$, and compute the sum of squared residuals $\mathsf{RSS}_{0i} = \sum_{t=1}^{T}(\hat{z}_{it}^2 - 1)^2$.
2. Regress $(\hat{z}_{it}^2 - 1)$ on $\hat{\mathbf{x}}_{it}^{hh}$, $\hat{\mathbf{x}}_{it}^{hg} + \hat{\mathbf{x}}_{it}^{gg}$ and $\hat{\mathbf{x}}_{it}^{hf}$, and obtain the sum of the squared residuals RSS_{1i} from the regression.
3. Finally, compute the non-robust test statistic

$$\mathsf{TR}_{nr,i}^2 = T\frac{\mathsf{RSS}_{0i} - \mathsf{RSS}_{1i}}{\mathsf{RSS}_{0i}} \tag{43}$$

   which has an asymptotic χ^2-distribution with $m-1$ degrees of freedom.

To obtain a multivariate test statistic, i.e., to test the multivariate null hypothesis $\mathsf{H}_0 : \boldsymbol{\vartheta}_f = (\boldsymbol{\vartheta}'_{f1}, \ldots, \boldsymbol{\vartheta}'_{fm})' = \mathbf{0}_{m(m-1)}$, repeat steps 1–3 for $i = 1, \ldots, m$ and compute

$$\mathsf{TR}_{nr}^2 = \sum_{i=1}^{m}\mathsf{TR}_{nr,i}^2. \tag{44}$$

Under H_0, the test statistic (44) is χ^2- distributed with $m(m-1)$ degrees of freedom.

A robust version of (43) to non-normal errors (see Wooldridge (1990, 1991)) may also be computed as follows:

1. Consistently estimate the MTV-GARCH$(1,1)$ model by maximization by parts and save the series of estimated volatility standardized residuals $\hat{z}_{it} = u_{it}/\sqrt{\hat{h}_{it}^0\hat{g}_{it}^0}$.
2. Regress $\hat{\mathbf{x}}_{it}^{hf}$ on $\hat{\mathbf{x}}_{it}^{hh}$ and $\hat{\mathbf{x}}_{it}^{hg} + \hat{\mathbf{x}}_{it}^{gg}$, and save the vector of residuals $\hat{\mathbf{r}}_{it}$ from the regression.
3. Regress $\mathbf{1}_T$ on $(\hat{z}_{it}^2 - 1)\hat{\mathbf{r}}_{it}$ and compute the sum of squared residuals RSS_i.
4. Compute the robust statistic

$$\mathsf{TR}_{r,i}^2 = T - \mathsf{RSS}_i \tag{45}$$

   which, under H_{0i}, is asymptotically χ^2-distributed with $m-1$ degrees of freedom.

Analogously to the non-robust case, to obtain the robust multivariate test statistic, repeat steps 1–4 and compute the χ^2-distributed statistic with $m(m-1)$ degrees of freedom as

$$\mathsf{TR}_r^2 = \sum_{i=1}^{m}\mathsf{TR}_{r,i}^2, \tag{46}$$

under the null hypothesis H_0. The finite sample properties of these test statistics shall now be studied.

3.1 Finite sample properties of the test

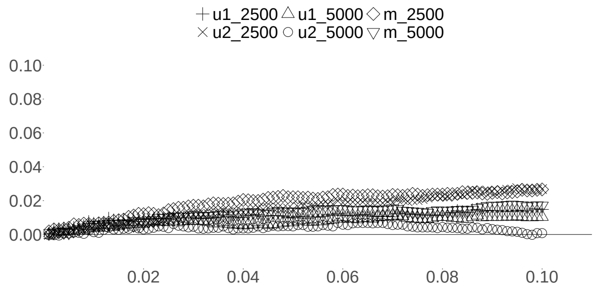
In this section, I study the finite sample properties of tests proposed for testing short-term volatility interactions. This is done by conducting a Monte Carlo experiment to study the size and power of the test statistics. These properties are investigated for different sample sizes of bivariate series, i.e., $T = 1000, 2500$ and 5000 observations. The number of replications is 5000 and, to minimise initialization effects, the first 1000 observations have been discarded. All computations are carried out using R 3.3.3 (R Core Team (2017)).

The volatility component of the bivariate series is generated according to the multiplicative specification (4)−(8) with $p = q = 1$. In the size and power simulations, the ARCH matrix $\mathbf{A}_1$ is assumed to be, respectively, diagonal and non-diagonal. For the correlation component, I consider constant conditional correlations. When H_{0i} holds for all $i = 1, \ldots, m$, the ARCH matrix is diagonal and the model is the MTV-CCC-GARCH$(1,1)$ model. Under the alternative, $\mathbf{A}_1$ is allowed to have non-negative off-diagonal elements and the model becomes the MTV-ECCC-GARCH$(1,1)$ model. The volatility and correlation structures used in the size simulations are reported in Table 2. These data generating processes (DGPs) are intended to have different levels of persistence and correlation. Persistence in volatility goes from moderate (0.90) to high (0.95) and very high (0.99). To study the impact of the correlation level on the test properties, the constant conditional correlation is moderate in all DGPs but DGP3, where it is higher. Finally, DGP2 and DGP4 are characterized by larger variations in the long-term volatilities compared to DGP1 and DGP3. Finally, the values of the parameters are chosen to be closer to the values usually found in empirical applications.

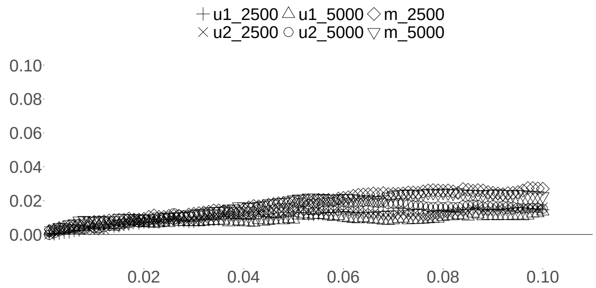
Table 2: Design of volatilities and correlation of the bivariate MTV-CCC-GARCH$(1,1)$ models used in the size simulations.

	Conditional variance subcomponent	*Deterministic subcomponent*	*Correlation*
DGP1	$h_{1t} = 0.10 + 0.10\phi^2_{1,t-1} + 0.85h_{1,t-1}$ $h_{2t} = 0.20 + 0.05\phi^2_{2,t-1} + 0.94h_{2,t-1}$	$g_{1t} = 1.2 - 0.05G_{1t}(5, 0.25; t/T)$ $g_{2t} = 1.2 + 0.05G_{2t}(10, 0.50; t/T)$	$\rho_{12} = 0.50$
DGP2	$h_{1t} = 0.10 + 0.10\phi^2_{1,t-1} + 0.85h_{1,t-1}$ $h_{2t} = 0.20 + 0.05\phi^2_{2,t-1} + 0.94h_{2,t-1}$	$g_{1t} = 1.0 - 0.05G_{1t}(5, 0.25; t/T)$ $g_{2t} = 2.5 + 0.05G_{2t}(10, 0.50; t/T)$	$\rho_{12} = 0.50$
DGP3	$h_{1t} = 0.10 + 0.10\phi^2_{1,t-1} + 0.85h_{1,t-1}$ $h_{2t} = 0.20 + 0.05\phi^2_{2,t-1} + 0.85h_{2,t-1}$	$g_{1t} = 1.2 - 0.05G_{1t}(5, 0.25; t/T)$ $g_{2t} = 1.2 + 0.05G_{2t}(10, 0.50; t/T)$	$\rho_{12} = 0.85$
DGP4	$h_{1t} = 0.10 + 0.10\phi^2_{1,t-1} + 0.85h_{1,t-1}$ $h_{2t} = 0.20 + 0.05\phi^2_{2,t-1} + 0.94h_{2,t-1}$	$g_{1t} = 1.2 - 0.75G_{1t}(5, 0.25; t/T)$ $g_{2t} = 1.2 + 1.50G_{2t}(10, 0.50; t/T)$	$\rho_{12} = 0.50$

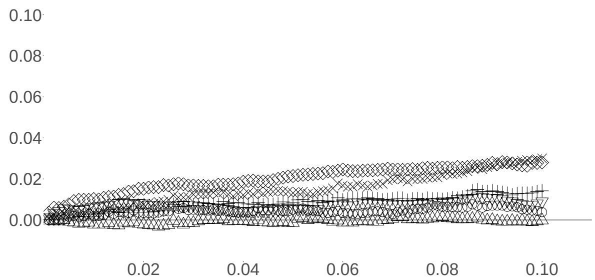
The non-robust and robust test statistics are obtained for each replication and the actual rejection frequency is computed and compared to different nominal significance levels ranging from 0.01 to 0.10. The size discrepancies in Figure 1 are plotted for the univariate and multivariate robust statistics given their superior performance compared to the non-robust ones. The size distortions are, in general, small, decreasing with the number of observations and increasing with the correlation. The difference between the empirical rejection frequency and the nominal level appears to be larger, the stronger is the time dependence in volatility. Results from additional simulated models, not reported for the sake of saving space, further support this conclusion. There is also evidence that the univariate robust test statistics outperform the multivariate robust ones for the sample sizes considered.



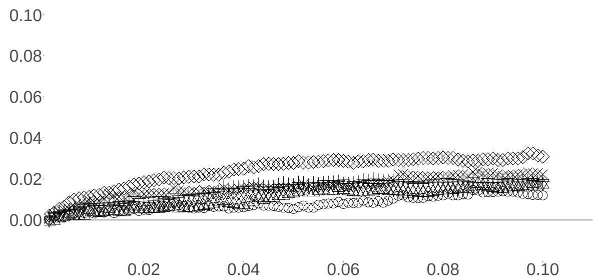
(a) DGP1



(b) DGP2



(c) DGP3



(d) DGP4

Figure 1: Size discrepancies plotted against the nominal significance level for different simulated MTV-CCC-GARCH$(1, 1)$ bivariate series. Results are shown for the univariate robust TR^2 (ui_T) and multivariate robust TR^2 (m_T) test statistics defined, respectively, in (45) and (46), $\mathrm{i} = 1, 2$ and $\mathrm{T} = 2500, 5000$.

The model used to generate the bivariate series in the power simulations is the MTV-ECCC-GARCH$(1, 1)$ model. The selected structures of the variance subcomponents and the correlation are summarised in Table 3. The CMCH effects are moderate for DGP 5, 7 and 8 ($\alpha_{12} = 0.01, \alpha_{21} 0.06$ in DGP 5 and $\alpha_{12} = \alpha_{21} = 0.05$ in DGP $7 - 8$) and low for series 2 in DGP 6 ($\alpha_{21} = 0.006$). Volatility persistence ranges from moderate in DGP $6{-}7$ to high in DGP 8 and very high in DGP 5. Correlation can be moderate as in DGP $5{-}6$ ($\rho_{12} = 0.50$), very high as in DGP 7 ($\rho_{12} = 0.90$) or low as in DGP 8 ($\rho_{12} = 0.30$).

The power curves are depicted in Figure 2. The power of the test tends to increase with the

Table 3: Design of volatilities and correlation of the bivariate MTV-ECCC-GARCH(1, 1) models used in the power simulations.

	Conditional variance subcomponent	*Deterministic subcomponent*
DGP5	$h^*_{1t} = 0.10 + 0.05\phi^2_{1,t-1} + 0.006\phi^2_{2,t-1} + 0.85h^*_{1,t-1}$	$g_{1t} = 1.50 - 0.05G_{1t}(5, 0.25)$
	$h^*_{2t} = 0.10 + 0.05\phi^2_{1,t-1} + 0.05\phi^2_{2,t-1} + 0.85h^*_{2,t-1}$	$g_{2t} = 1.00 + 0.10G_{2t}(10, 0.50)$
DGP6	$h^*_{1t} = 0.05 + 0.05\phi^2_{1,t-1} + 0.05\phi^2_{2,t-1} + 0.80h^*_{1,t-1}$	$g_{1t} = 0.95 + 0.05G_{1t}(5, 0.50)$
	$h^*_{2t} = 0.05 + 0.05\phi^2_{1,t-1} + 0.05\phi^2_{2,t-1} + 0.80h^*_{2,t-1}$	$g_{2t} = 1.05 - 0.03G_{2t}(5, 0.50)$
DGP7	$h^*_{1t} = 0.05 + 0.05\phi^2_{1,t-1} + 0.05\phi^2_{2,t-1} + 0.80h^*_{1,t-1}$	$g_{1t} = 0.95 + 0.05G_{1t}(5, 0.50)$
	$h^*_{2t} = 0.05 + 0.05\phi^2_{1,t-1} + 0.05\phi^2_{2,t-1} + 0.80h^*_{2,t-1}$	$g_{2t} = 1.05 - 0.03G_{2t}(5, 0.50)$
DGP8	$h^*_{1t} = 0.05 + 0.05\phi^2_{1,t-1} + 0.05\phi^2_{2,t-1} + 0.85h^*_{1,t-1}$	$g_{1t} = 0.95 + 0.05G_{1t}(5, 0.50)$
	$h^*_{2t} = 0.05 + 0.05\phi^2_{1,t-1} + 0.05\phi^2_{2,t-1} + 0.85h^*_{2,t-1}$	$g_{2t} = 1.05 - 0.03G_{2t}(5, 0.50)$

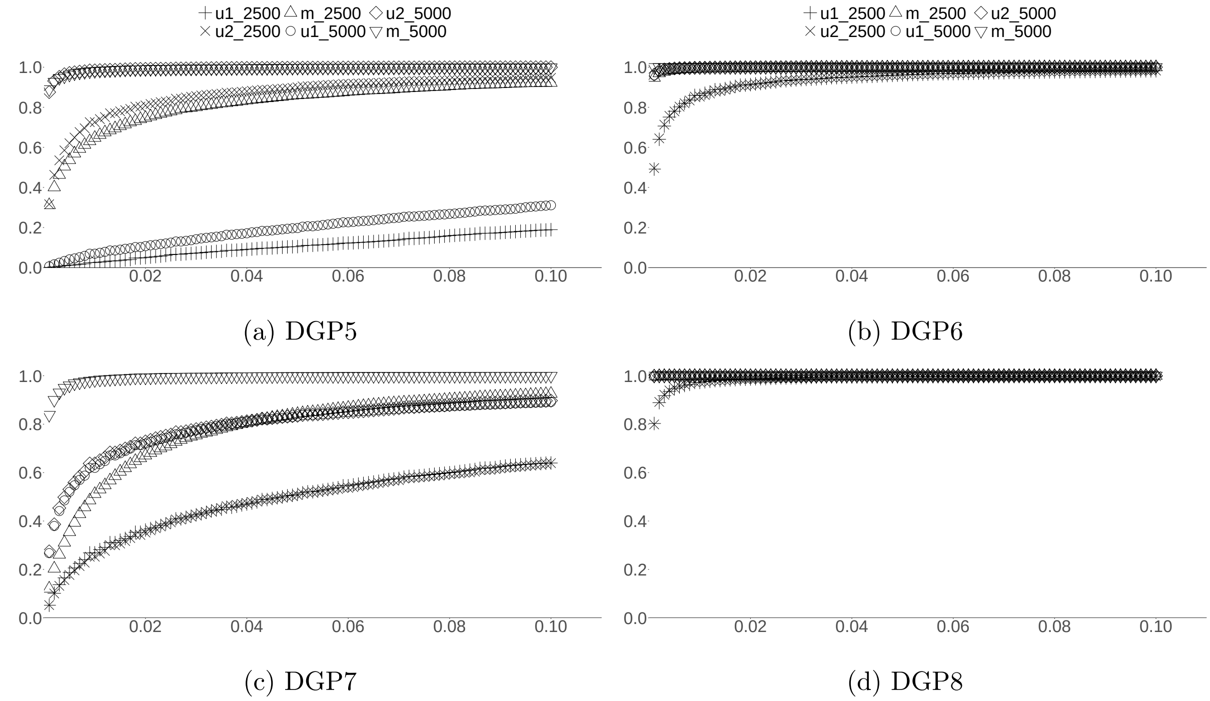
Correlation
DGP5: 0.50 DGP6: 0.50 DGP7: 0.90 DGP8: 0.30

number of observations and stronger CMCH effects (power is relatively higher for series 2 in DGP 5) and decrease when persistence in volatility is very high (DGP 5) or when correlation is very high (DGP 7).

4 Modelling exchange rate comovements

An application of the test for short-term volatility interactions and the estimation strategy to real data is provided in this section. Previous empirical applications include Diebold & Nerlove (1989) who applied a multivariate ARCH model to study the comovements in volatilities of exchange rates. In their latent factor ARCH model, univariate volatility models are used to help specifying the multivariate model. Heteroskedasticity is captured by the ARCH structure whilst the "common volatility movements" are captured by the factor structure. On the dynamics of volatility comovements of exchange rates before and after the introduction of the euro, Antonakakis (2012) found the euro to be a net transmitter of volatility whereas the British pound moved as a net volatility receiver. They also concluded that comovements in exchange rate volatilities can change largely over time and are positively associated with economic crises, including financial crises.

The main idea is to first analyse structural changes in the long-term volatility processes and then model the short-term volatility interactions using the equation by equation procedure described in section 2.3. Preceding the modelling of the short-term volatility interactions, I test for their presence by using the LM-type test derived in section 3. I use daily observations of four major spot exchange rates against the euro, among the most traded currencies, from the Bank of England Database. The return series are 100 times the log-differences of the daily exchange rates of the U.S. dollar (EUR/USD), the Japanese yen (EUR/JPY), the British pound (EUR/GBP)



(a) DGP5 (b) DGP6

(c) DGP7 (d) DGP8

Figure 2: Power curves for different simulated MTV-ECCC-GARCH(1, 1) bivariate series. Results are shown for the univariate (ui_T), $i = 1, 2$, and multivariate (m_T) robust TR^2 test statistics defined, respectively, in (45) and (46) where the sample size $T = 2500, 5000$.

and the Australian dollar (EUR/AUD) with respect to the euro (EUR)[6]. The observation period goes from the 5th of January, 1999 until the 17th September, 2010, yielding a total of 2998 observations. The dataset is exactly the same as in Francq & Zakoïan (2012), who analysed the volatility comovements of the U.S. dollar and the Japanese yen exchange rates for the total period and for three sub-periods. Notice that, to enrich the dataset, two additional exchange rate series are considered in this empirical application. The authors used the constant conditional correlation asymmetric (CCC-A) GARCH model, which can be regarded as an extension of the ECCC-GARCH model by including (cross-market) leverage effects. According to their results, volatility parameters appear to change from one sub-period to another, indicating nonstationarity in the volatility processes and structural changes in the volatility spillovers. This leads us to conclude that the model under usage in Francq & Zakoïan (2012) may not be the most appropriate to fit the data.

The plots with the daily return series are depicted in Figure 3. For the JPY, GBP and AUD, larger volatility at the beginning and ending of the observation period and lower volatility for the period in between is observed. For the USD, identifying volatility regimes by visual inspection of the series is not so clear even though some extreme returns are also observed in the first and

[6]The exchange rates are defined as one unit of EUR in terms of other currencies. To simplify the notation, I denote the exchanges rates simply as USD, JPY, GBP and AUD.

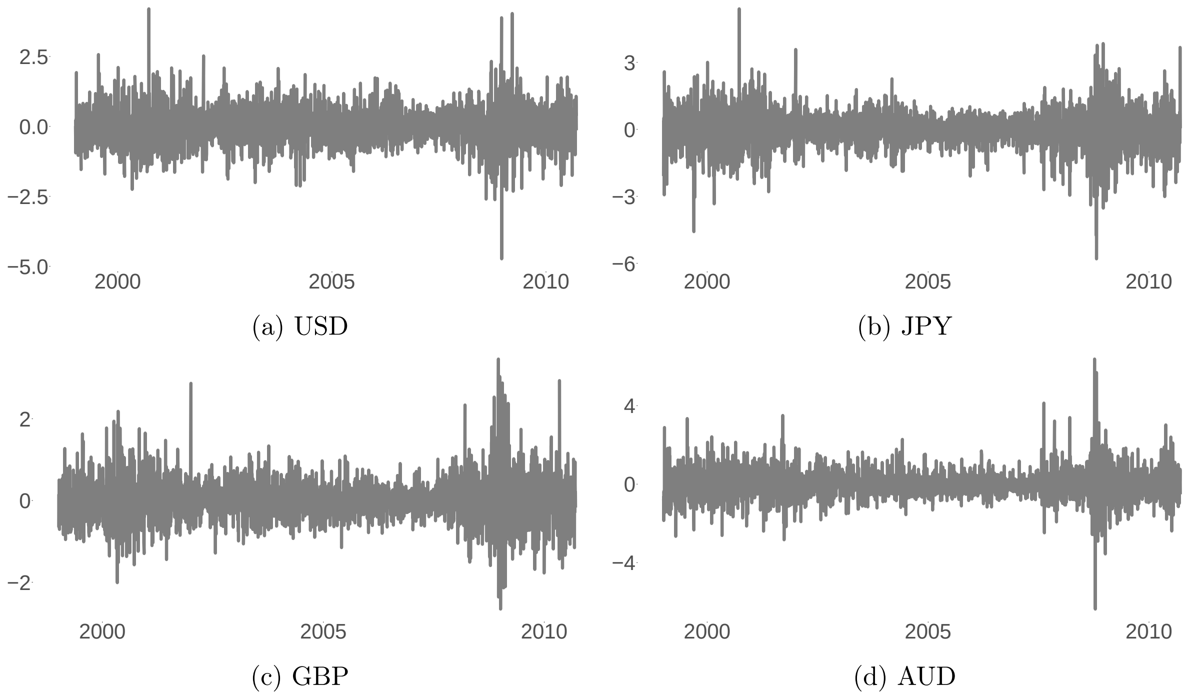


Figure 3: Daily returns on the exchange rates of the U.S. dollar (USD), the Japanese yen (JPY), the British pound (GBP) and the Australian dollar (AUD) with respect to the euro from January 15, 1999 until September 17, 2010.

Table 4: Descriptive statistics.

	Min.	Mean	Max.	Std. Dev.	Krt.	Rob. Krt.	Skw.	Rob. Skw.
USD	−4.735	0.003	4.204	0.667	2.767	0.173	0.108	0.009
JPY	−5.800	−0.006	5.396	0.814	3.949	0.121	−0.235	−0.043
GBP	−2.657	0.005	3.461	0.515	3.986	0.172	0.463	0.004
AUD	−6.370	−0.011	6.377	0.724	8.043	0.158	0.530	0.041

fourth quartiles.

The descriptive statistics, including the non-robust and robust measures of the kurtosis and skewness are reported in Table 4. The diagnostic tests for time dependence, namely the non-robust (Q(5)) and the corrected (rob.Q(5)) portmanteau test in the presence of ARCH effects proposed by Francq & Zakoïan (2009) up to order 5, and the test for ARCH effects up to order 5 (ARCH(5)) of Engle (1982), are shown in Table 5. By analysing the summary statistics, the daily returns have approximately zero mean but other stylized features of daily returns, such as leptokurtosis and negative skewness, are not so clear when considering their robust counterparts. According to the robust autocorrelation test, returns are serially independent but the same is not true for the variances since the heteroskedasticity tests provide clear evidence that the squared returns are time dependent. Finally, the Jarque-Bera tests for the composite hypothesis of normality shown in the last column of Table 5 indicate that the return series are non-normal.

Table 5: Results (p-values in parentheses) from the non-robust (Q(5)) and the robust (rob. Q(5)) autocorrelation tests, the heteroskedasticity (ARCH(5)) tests, and the Jarque-Bera normality tests (JB).

	non-rob. Q(5)	rob. Q(5)	ARCH(5)	JB
USD	3.751	3.185	39.71	962.1
	(0.586)	(0.671)	(0.000)	(0.000)
JPY	2.512	1.324	89.01	1976
	(0.775)	(0.932)	(0.000)	(0.000)
GBP	16.63	7.101	100.2	2092
	(0.005)	(0.213)	(0.000)	(0.000)
AUD	19.20	3.261	132.4	8222
	(0.002)	(0.660)	(0.000)	(0.000)

I follow the modelling strategy described in section 2.3. By observing that a first-order GARCH model is sufficient to capture the heteroskedastic behaviour of the exchange rate return series, I proceed to testing the null hypothesis of constant unconditional variance against a smoothly changing baseline volatility structure. The results from the robust test of the GARCH$(1, 1)$ model against the MTV-GARCH$(1, 1)$ model is presented in the upper panel of Table 6. Constant long-term volatility is rejected for the JPY, GBP and AUD log-return series. According to the decision rule for selecting the number of location parameters, $r_i = 2$ for $i =$ JPY, GBP and AUD[7]. After accounting for the structural changes in the baseline volatilities, I test the hypothesis of an additively misspecified conditional variance subcomponent among these return series. For that purpose I am considering bivariate models such that I test the CMCH effect from a specific exchange rate innovation series to another series. The test results for the presence of short-term volatility interactions can be found in the middle panel of Table 6. They indicate that the volatility structure of GBP is affected by, not only its own past information (known as the ARCH effect), but also by the past squared innovations of all remaining exchange rates (the CMCH effects). There is also evidence that the volatility of AUD is affected by the past squared innovations of JPY.

For the USD return series, I proceed to testing the presence of volatility spillovers by using the multivariate tests proposed by Pedersen (2017) and Nakatani & Teräsvirta (2009). These tests refer to the class of conditional correlation GARCH models where the volatility component is modelled as a standard vector GARCH process (without assuming any diagonality of the ARCH and GARCH matrices). Under the null hypothesis, there are no interactions between the volatility processes and so the ARCH and GARCH matrices are assumed as diagonal.

The test statistics proposed by Pedersen (2017) can be viewed as "corrected" versions of the test introduced by Nakatani & Teräsvirta (2009) when the true parameter vector is at the boundary of the parameter space. Results from the test statistic using auxiliary regressions proposed by Nakatani & Teräsvirta (2010) are reported in Table 7 and denoted by non-rob. $\mathsf{TR}^2_{\mathrm{ECCC}}$ and rob. $\mathsf{TR}^2_{\mathrm{ECCC}}$ for, respectively, the non-robust and robust TR^2 volatility spillover test

[7]The number of location parameters is chosen according to the lowest $p-$value when testing the sequence of nested hypotheses.

Table 6: Results from the robust test of volatility parameter constancy and decision rule for selecting the number of location parameters in the transition function of the multiplicative TV-GARCH$(1,1)$ model (upper panel) and two misspecification tests, one where h_{it} is additively misspecified (middle panel) in the TV-GARCH$(1,1)$ model and the other for an additional transition in the g_{it} component (lower panel) of the (extended) TV-GARCH$(1,1)$ model. Boldface indicates rejection of the null hypothesis at the 5% significance level.

	USD		JPY		GBP		AUD	
	Statistic	p-value	Statistic	p-value	Statistic	p-value	Statistic	p-value
				Single transition				
LM_0	1.732	0.785	7.693	0.103	9.839	**0.043**	6.822	0.146
LM_{03}	0.123	0.725	0.726	0.394	0.660	0.417	0.052	0.820
LM_{02}	1.330	0.249	4.323	**0.038**	7.131	**0.008**	4.303	**0.038**
LM_{01}	0.313	0.855	1.960	0.375	2.003	0.367	1.775	0.412
				Volatility interactions				
USD			2.380	0.123	7.983	**0.005**	2.604	0.107
JPY	–	–			5.523	**0.019**	6.936	**0.008**
GBP	–	–	3.326	0.068			1.565	0.211
AUD	–	–	2.551	0.110	4.271	**0.039**		
				Double transition				
LM_0	–	–	7.512	0.111	20.72	**0.000**	9.564	**0.048**
LM_{03}	–	–	2.447	0.118	8.816	**0.003**	1.393	0.238
LM_{02}	–	–	1.489	0.222	1.180	0.277	0.232	0.630
LM_{01}	–	–	2.184	0.336	6.892	**0.032**	8.009	**0.018**

Table 7: Results from the bivariate tests of volatility spillovers proposed by Pedersen (2017) (Wald, QLR, $\mathsf{LM_D}$ and LM) and Nakatani & Teräsvirta (2010) (non-rob. and rob. $\mathsf{TR}^2_{\text{ECCC}}$). The null hypothesis of a diagonal CCC-GARCH model is tested against the alternative of an ECCC-GARCH model. Boldface indicates rejection of the null hypothesis at the 5% significance level.

	USD−JPY		USD−GBP		USD−AUD	
	Statistic	*p*-value	Statistic	*p*-value	Statistic	*p*-value
Wald	3.152	0.398	29.11	**0.023**	23.24	**0.040**
QLR	3.529	0.372	43.22	**0.006**	28.56	**0.023**
$\mathsf{LM_D}$	3.311	0.387	36.81	**0.011**	26.84	**0.028**
LM	1.620	0.805	15.51	**0.004**	8.251	0.083
non-rob. $\mathsf{TR}^2_{\text{ECCC}}$	3.096	0.542	12.73	**0.013**	6.502	0.165
rob. $\mathsf{TR}^2_{\text{ECCC}}$	3.861	0.425	12.48	**0.014**	6.811	0.146

statistics. Results from the Wald, Quasi Likelihood Ratio (QLR), Directed Lagrange Multiplier ($\mathsf{LM_D}$) and standard Lagrange Multiplier (LM) test statistics discussed in Pedersen (2017) are also reported. For computing the $p-$values for the standard LM-type statistics, a $\chi^2_{2m(m-1)}$ distribution is assumed with $2m(m-1)$ denoting the number of degrees of freedom. For the Wald, QLR, and $\mathsf{LM_D}$ statistics, the $p-$values are based on the estimated critical values obtained from their limiting distributions. For more details, see the notes in Table 5 and Section 4.2 in Pedersen (2017). The presence of volatility spillovers is tested for the pairwise USD−JPY, USD−GBP and USD−AUD. By looking at the $p-$values, I am very confident to state that no volatility spillovers exist between the pairwise USD−JPY and USD−AUD during the period analysed. This may explain the very small estimated off-diagonal elements of the ARCH and GARCH matrices in Francq & Zakoïan (2012) for USD−JPY. I find however evidence in favour of volatility spillovers for USD−GBP where the diagonality of the ARCH and GARCH matrices is strongly rejected by all test statistics, and for USD−AUD.

When using multivariate test statistics, I am not able to infer about the direction of the cross-market volatility effects. Thus, I complement the results with an equation by equation version of the $\mathsf{TR}^2_{\text{ECCC}}$ test statistics in Nakatani & Teräsvirta (2010) (for the sake of saving space, these results are not reported). According to the results, the volatility of the GBP exchange rate returns is not only affected but also affects the volatility on the USD exchange rate returns. For the pairwise USD−AUD the univariate test results fail to reject the null hypothesis in both directions. Fitting different GARCH models to the USD return series, namely the standard GARCH and the augmented GARCH with GBP and with GBP and AUD as covariates, I observe that the model that minimizes the Bayesian information criterion (BIC) of Schwarz (1978) is the GARCH model with GBP as covariate.

Combining the previous statistical test results, I assume the volatility processes follow the GARCH$(1,1)$ model for USD with GBP as covariate, the MTV-GARCH$(1,1)$ model for JPY, the MTV-GARCH$(1,1)$ model for GBP with USD, JPY and AUD as covariates and, finally, the MTV-GARCH$(1,1)$ model for AUD with JPY as covariate. After fitting these models to the

Table 8: Estimated parameters in the deterministic subcomponent. Estimates are obtained using the equation by equation estimator and the robust standard errors (in parentheses) are computed based on numerical derivatives.

	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\gamma}_1$	$\hat{c}_{11}$	$\hat{c}_{12}$	$\hat{\delta}_2$	$\hat{\gamma}_2$	$\hat{c}_{21}$	$\hat{c}_{22}$	$\hat{c}_{23}$
JPY	0.275 (–)	0.862 (0.146)	4.623 (–)	0.215 (0.013)	0.766 (0.020)					
GBP	5×10^{-5} (–)	0.296 (0.063)	4.743 (–)	0.203 (0.039)	0.768 (0.007)	0.237 (0.058)	3.742 (–)	0.169 (0.098)	0.570 (0.102)	0.916 (0.128)
AUD	5×10^{-5} (–)	0.811 (0.125)	2.525 (–)	0.345 (0.051)	0.987 (0.048)	0.440 (0.101)	5.495 (–)	0.730 (0.003)		

Table 9: Estimated parameters in the conditional variance subcomponent. Persistence is reported for the CCC-GARCH model (1), the MTV-CCC-GARCH model (2) and the MTV-ECCC-GARCH model with one transition function (3) and two transition functions (4). Estimates are obtained using the equation by equation estimator and the robust standard errors (in parentheses) are computed based on numerical derivatives.

	$\hat{\omega}$	$\hat{\alpha}_{\text{USD}}$	$\hat{\alpha}_{\text{JPY}}$	$\hat{\alpha}_{\text{GBP}}$	$\hat{\alpha}_{\text{AUD}}$	$\hat{\beta}$	(4)	(3)	(2)	(1)
USD	0.002 (0.001)	0.027 (0.005)		0.012 (0.006)		0.962 (0.007)	0.995	0.995	0.997	0.997
JPY	0.034 (0.013)		0.067 (0.016)			0.899 (0.026)	0.966	0.966	0.966	0.997
GBP	0.038 (0.017)	0.033 (0.020)	0.015 (0.014)	0.045 (0.013)	0.004 (0.004)	0.883 (0.039)	0.922	0.935	0.974	0.995
AUD	0.073 (0.061)		0.034 (0.027)		0.073 (0.034)	0.816 (0.117)	0.889	0.929	0.957	0.993

data and before proceeding to the estimation of the correlation parameters, I test the possibility that g_{it} is also misspecified. Specifically, I check the necessity of an additional transition to capture the daily variation in the long-term volatility of JPY, GBP and AUD. The results from these misspecification tests are reported in the lower panel of Table 6. The null hypothesis of a single transition is rejected for GBP and AUD meaning that an additional transition function with, respectively, three and one location parameters are necessary for modelling the baseline volatility of these return series. The final estimation results for the volatility component of the MTV-ECCC-GARCH$(1, 1)$ model are shown in Tables 8 and 9 for, respectively, the deterministic and conditional variance subcomponents.

In Table 8, I present the estimated volatility and transition parameters in the deterministic subcomponent g_{it}. The estimated values of the speed of transition are moderate meaning the transitions from one volatility state to the other are relatively smooth and slow. The similarity of the locations of transition in the first transition function reveals a common long-term variability in the volatilities of the JPY and GBP exchange rates. This result can also be seen in Figure 4 by the dynamics of the estimated deterministic subcomponents (in blue) for JPY and GBP. The baseline volatility starts to decrease in 2001 and remains low for a long period until the end of 2007, when it starts increasing again towards the initial level. Higher volatility appears to be related with crisis periods, namely the dot-com bubble and the sub-prime crisis and subsequent global financial crisis. The period in between is characterized as a calm period with low and stable

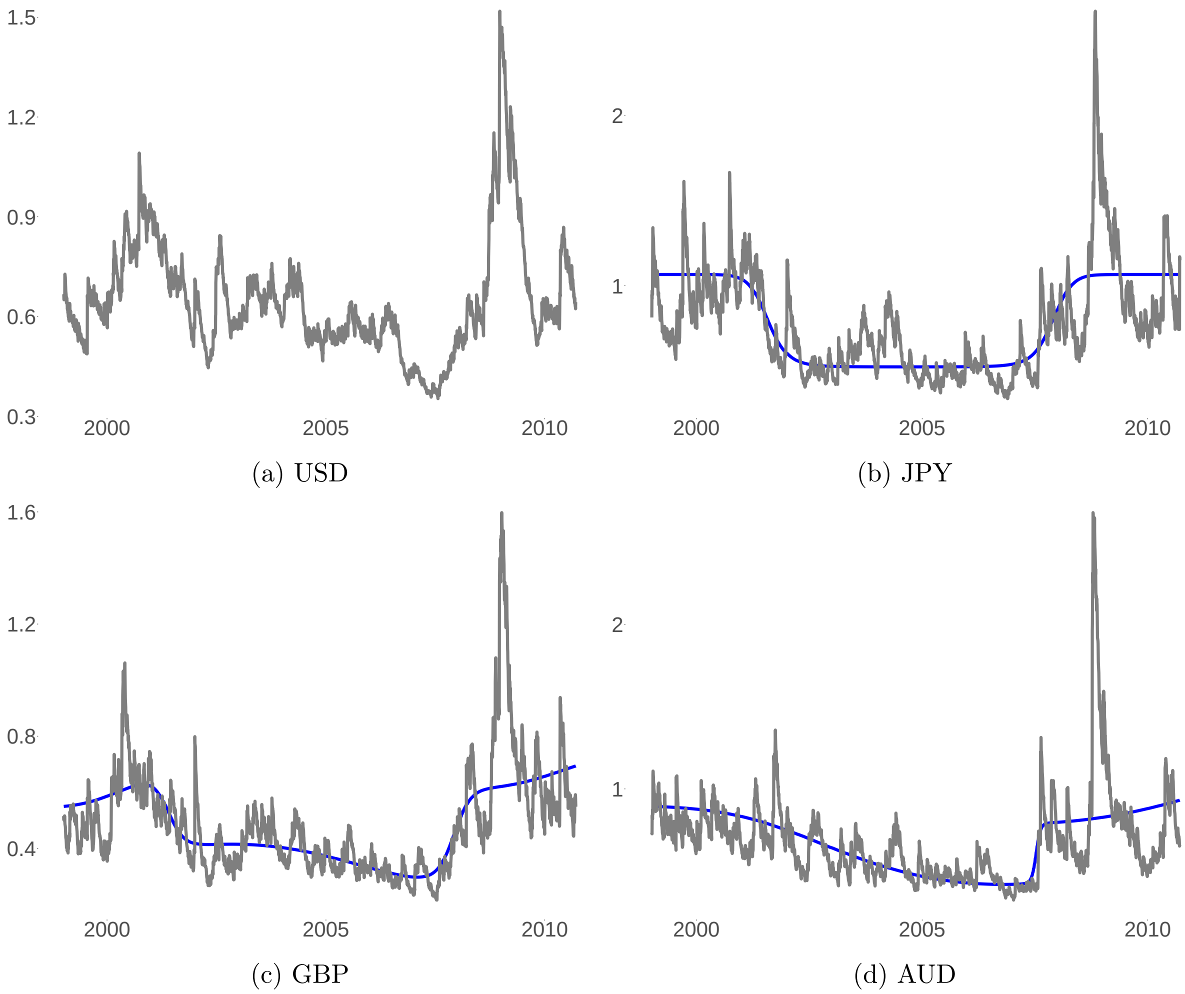


Figure 4: Volatilities from estimated standard GARCH$(1, 1)$ models (grey) and the estimated long-term volatilities defined by the g_{it} subcomponent (blue) for JPY, GBP and AUD.

long-term volatility. A somehow similar pattern is found for AUD where the first transition is clearly smoother than the second one.

To evaluate how the slow (low-frequency) movements in the baseline volatility compare to the quick (high-frequency) movements obtained from a standard GARCH process, I add the estimated volatilities using a GARCH$(1, 1)$ model to each panel. It is interesting to see how the baseline volatility follows the dynamics of the GARCH process as a smoothed conditional mean. The plots for the estimated volatility of the USD and the estimated short-term volatilities denoted as h_{it} (weakly stationary GARCH process) for JPY and h^*_{it} (weakly stationary GARCH process augmented with CMCH effects) for GBP and AUD are shown in Figure 5.

For comparison, I also report in Table 9 the persistence level for different estimated GARCH models, measured by the sum of the estimated ARCH and GARCH coefficients for standard GARCH models and by the eigenvalues (in decreasing order) of $\mathbf{A}_1 + \mathbf{B}_1$ for extended GARCH models. From right to left, I show the volatility persistence estimated from the CCC-GARCH model (1), the MTV-CCC-GARCH model with one transition function and no volatility interactions (2), the MTV-ECCC-GARCH model with one transition function and the identified volatility interactions (3), and finally, the MTV-ECCC-GARCH model with two transition functions and the identified volatility interactions (4). The transition functions are only estimated for the identified nonstationary volatilities, namely JPY, GBP and AUD. The model presented in column (4) is the final model fitted to the data. Based on the results, I can conclude that persistence can be reduced by including the CMCH effects as covariates and/or rescaling the innovation series by the baseline volatility. Regarding the off-diagonal elements of the ARCH matrix, i.e., the cross-market volatility interactions, I find a bidirectional statistically significant effect between the volatilities of USD and GBP return series.

After estimating the volatility component, I proceed to the estimation of the correlation component. For each bivariate series, the estimated dynamic conditional correlations (grey) whose structure is defined in (10)−(11) and the estimated deterministic unconditional correlations (blue) defined in (12) are depicted in Figure 6. For comparison, I also plot the constant conditional correlations (red). The dynamic conditional correlations appear to fluctuate around the slow unconditional correlations. In general, a decreasing trend in the correlations of the exchange rate returns is observed meaning that the comovements between them tend to become weaker over time.

5 Concluding remarks

I present a new conditional correlation GARCH model with nonstationary GARCH equations for modelling the conditional volatilities while considering alternative structures for the correlations. By multiplicatively decomposing the variance equations into the conditional variance and the deterministic time-varying subcomponents, the model is able to capture, respectively, short- and long-term movements in volatilities. In multivariate series, interactions between the volatility processes usually occur meaning that shocks in one market affect not only its own volatility but also other markets' volatilities. As a misspecification test of the MTV-GARCH model, I propose a new test for the presence of short-term volatility interactions. Specifically, I want to check the necessity for including past cross-market series information in the conditional variance subcomponent. A Monte Carlo experiment shows that the robust test using auxiliary regressions has good size and power in finite samples.

An empirical application to major exchange rates, provides evidence for the presence of volatility interactions between the U.S. dollar, the Japanese yen, the British pound and the Australian dollar exchange rate returns. In particular, the British pound appears to move as a volatility receiver (a similar result is found by Antonakakis (2012)) since, according to the statistical test for short-term volatility interactions, shocks in all the remaining exchange rates seem to affect the volatility of the GBP exchange rate. As usually observed in financial applications, volatility tends to be higher during crisis periods and lower in tranquil times. Including non-stationarity and/or CMCH effects in the variance equations appears to reduce volatility persistence.

For inference, I am assuming the estimators are consistent and asymptotically normal. Asymptotic properties are known for some models nested in the general model proposed in this paper but they cannot be directly extended. I leave the asymptotic theory for future work. Another challenge when modelling volatilities and correlations, is to evaluate how well the model explains stylized facts of financial data and how good it is for making predictions. Misspecification tests may be helpful to give insight on what good is a volatility model. The conditional variance subcomponent may still be misspecified and so should also include, besides the CMCH effects, asymmetric ARCH effects or higher ARCH and GARCH orders. The need to include another transition function, as seen in the empirical application, is also a relevant hypothesis to test when the deterministic subcomponent is also misspecified. These statistical tests are also seen as posterior improvements or extensions to the paper.

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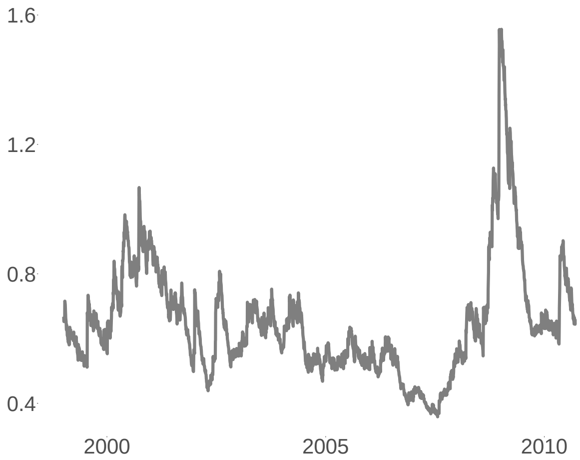
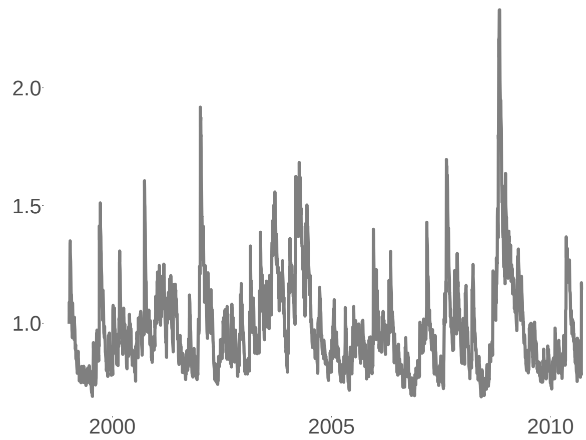
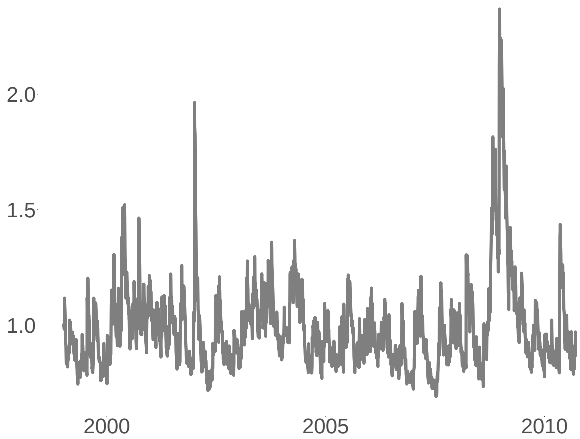


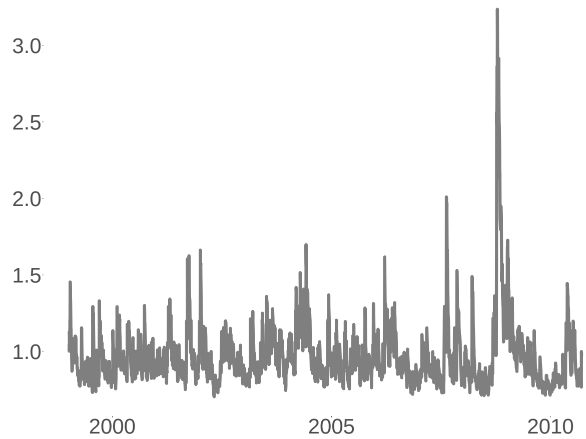
(a) USD: GARCH with GBP



(b) JPY: MTV-GARCH



(c) GBP: MTV-GARCH with USD, JPY and AUD



(d) AUD: MTV-GARCH with JPY

Figure 5: Volatilities from the estimated GARCH(1, 1) model augmented with CMCH effects for USD, the estimated subcomponent h_t in the MTV-GARCH(1, 1) model for JPY, and the estimated subcomponent h_t^* in the MTV-GARCH(1, 1) model augmented with CMCH effects for GBP and AUD.

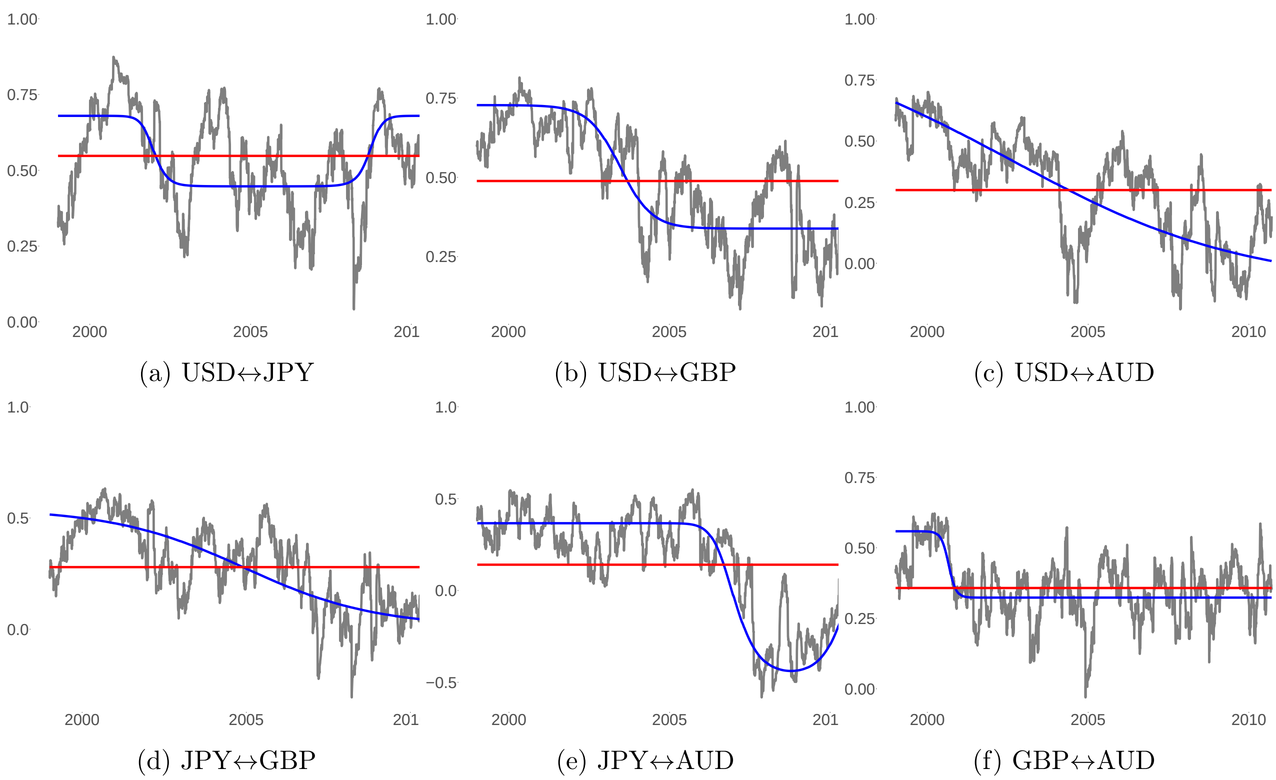


Figure 6: The estimated constant conditional correlations (red), dynamic conditional correlations (grey) and the time-varying unconditional correlations (blue).