Long-Range Dependent Continuous-Time Financial Econometric Models: Theory and Practice

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Abstract

Data with long–range dependency can be modelled with continuous–time fractional stochastic processes. The process studied in this paper is an extension of the classical Ornstein–Ulhenbeck process, containing an extra parameter related to the Hurst index. A semiparametric estimation procedure is posed and tested within. A continuous–time version of the Gauss–Whittle contrast function, measures the discrepancy between the data periodogram and its spectral density. As a special application, the proposed estimation procedure is applied to a class of fractional stochastic volatility processes. In addition, the long–range dependency of the returns for the S&P 500 and the T–Bill rate is studied.

KEYWORDS: Continuous–time model, diffusion process, long–range dependence, stochastic volatility.
JEL Subject Classification: C13; G13

1. INTRODUCTION

Since the publication of Merton (1969), continuous–time processes have been closely associated with finance. Thus, the variation of a security price is roughly calculated as the sum of its multiple variations during the given time period. The main assumption of the continuous–time theory is that these security price variations happen over infinitesimal intervals of time. Perhaps the most popular application of this theory has been the contribution to option pricing by Black and Scholes (1973) and Merton (1973), in which the option price problem is reduced to finding the solution to a partial differential equation. In general, any contingent claim that has an unpredictable outcome in the future can be modelled in continuous–time by a Brownian motion process. Consider a stochastic differential equation (SDE) of the form,

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t), \tag{1}$$

where $\mu(X(t))$ is the drift function and $\sigma(X(t))$ is the volatility function of the process. Analytical solutions to these models are not always available. This motivates the development of numerical and estimation techniques. For instance, Platen (1999) and Kloeden and Platen (1999) extend numerical methods used to find approximations of solutions of ordinary differential equations to find approximations of solutions of SDEs. At the same time, there has been an important development of estimation techniques for continuous–time models which can be grouped into: maximum likelihood methods, generalised method of moments, simulated method of moments, efficient method of moments, nonparametric approaches and methods based on empirical characteristic functions (see Sundaresan 2000).

There is a vast list of references related to developments on the short–term interest rate as a stochastic diffusion process. Vacisek (1977) proposes a model of type (1) with the variance independent of the interest rate. Cox, Ingersoll and Ross (1985) extend this case to a model where the variance is proportional to the interest rate. Such a model is termed as the well–known CIR process. Hull and White (1987) amongst others, study the logarithm of the stochastic volatility (SV) as an Ornstein–Uhlenbeck process. Andersen and Lund (1997) extend the CIR model to associate the spot interest rate with stochastic volatility process through estimating the parameters with the efficient method of moments. Other closely related studies include Aït–Sahalia (1996, 1999), Gao and King (2004), and Hong and Li (2004).

In recent years, there have been both theoretical and applied studies for dealing with cases where data may exhibit long–range dependence (LRD) (see Ding, Granger and Engle 1993; Robinson 1994, 1999; Baillie and King 1996; Comte and Renault 1996, 1998; Ding and Granger 1996; Anh and Heyde 1999; Heyde 1999; Deo and Hurvich 2001; Gao, *et al.* 2001; Gao, Anh and Heyde 2002; Gao 2004; and others). For the case of continuous–time models, Comte and Renault (1996) prove that classical SDE models can be extended to embrace LRD models. They also show that how this extension is more suitable in a continuous–time framework than in a discrete time framework. The main characteristic of these extended models is the substitution of the classical Brownian motion by the so-called fractional Brownian motion of the form $B_\beta(t) = \int_0^t \frac{(t-s)^\beta}{\Gamma(1+\beta)} dB(s)$, where $B(t)$ is the standard Brownian motion and $\Gamma(x)$ is the usual Γ function. A Hurst index, H, with values in the interval $(\frac{1}{2}, 1)$ quantifies that the data exhibit LRD. The parameter β is related to the Hurst index through the expression $H = \beta + \frac{1}{2}$ (see Beran 1994, p.52–53), therefore β is defined as the LRD parameter when $0 < \beta < \frac{1}{2}$. For $0 < \beta < \frac{1}{2}$ (i.e., $\frac{1}{2} < H < 1$) the process is said to have LRD, for $d = 0$ (i.e., $H = \frac{1}{2}$) the observations are uncorrelated, and for $-\frac{1}{2} < d < 0$ (i.e., $0 < H < \frac{1}{2}$) the process is said to have short–range dependence (SRD). In practice, Ding, Granger and Engle (1993) suggest that financial aggregate data, such as the absolute return for Standard & Poor 500 Stock Price Index, may display LRD. That is, transformations of the autocorrelation function for large lags have non–negligible values.

Throughout this paper, the models used are determined by the continuous–time fractional stochastic differential equation of the form

$$dX(t) = -\alpha X(t)dt + \sigma dB_\beta(t),\ X(0) = 0,\ t \in (0, \infty). \tag{2}$$

The solutions to this diffusion equation are processes with a spectral density defined by

$$\phi(\omega) = \phi(\omega, \theta) = \frac{\sigma^2}{\Gamma^2(1+\beta)} \frac{1}{|\omega|^{2\beta}} \frac{1}{\omega^2 + \alpha^2}, \tag{3}$$

where $\theta \in \Theta = \left\{\theta = (\alpha, \beta, \sigma) : \alpha > 0, -\frac{1}{2} < \beta < \frac{1}{2}, \sigma > 0\right\}$. In this equation, α is the drift parameter, σ is the volatility parameter and $B_\beta(t)$ is as defined before. The well–known short–term interest rate model proposed by Vasicek (1977) is a special case of model (2) with $\beta = 0$. The spectral density described in equation (3) corresponds to that of an Ornstein–Ulhenbeck process of the form (2) driven by fractional Brownian motion with Hurst index $H = \beta + \frac{1}{2}$.

The solutions $X(t)$ of (2) are given by

$$X(t) = \int_0^t A(t-s)dB(s) \tag{4}$$

with $A(x) = \frac{\sigma}{\Gamma(1+\beta)}\left(x^\beta - \alpha\int_0^x e^{-\alpha(x-u)}u^\beta du\right)$. It follows from equation (4) that $X(t)$ belongs to a family of non–stationary Gaussian processes. It is known though, that a stationary version, $Y(t)$, of $X(t)$ can be found as follows:

$$Y(t) = \int_{-\infty}^t A(t-s)dB(s). \tag{5}$$

Comte and Renault (1998) were among the first to study the estimation of the LRD parameter β involved in model (2). In their study, an approximation to the solution given by equation (4) is found using a pathwise fractional integration method. As an application, they use this method to estimate β as a parameter of a fractional stochastic volatility (FSV) model of the form

$$d\ln(S(t)) = v(t)dB(t), \tag{6}$$

$$d\ln(v(t)) = -\alpha\ln(v(t))dt + \sigma dB_\beta(t), \tag{7}$$

where the parameters and Brownian motion are defined as above.

As can be seen, models (2) and (7) are also determined by the drift parameter, α, and the volatility parameter, σ. More recently, Gao (2004) proposes an estimation procedure for the case where $\theta \in \Theta_1 = \left\{\theta = (\alpha, \beta, \sigma) : \alpha > 0, 0 < \beta < \frac{1}{2}, \sigma > 0\right\}$ for model (2) and establishes some asymptotic properties for the proposed estimation procedure.

To check whether the estimation procedure proposed in Gao (2004) remains applicable to some well–known financial data, such as the S&P 500 Stock Price Index, we choose several different sections of the data and then check whether all the chosen sections of the data exhibit LRD. Our empirical studies show that the estimated values of β based on some sections of the data appear to be within the interval of $\left(0, \frac{1}{2}\right)$ while the resulting estimated values of β based on other sections of the data look negative. This motivates the extension of the proposed estimation procedure to the case where $\theta \in \Theta = \left\{\theta = (\alpha, \beta, \sigma) : \alpha > 0, -\frac{1}{2} < \beta < \frac{1}{2}, \sigma > 0\right\}$.

The main contribution of the current paper thus includes: (i) methodologically, it proposes a general estimation procedure to deal with cases where the process may display possible LRD or SRD; (ii) our comprehensive simulation studies show that the proposed estimation procedure works well numerically not only for the LRD parameter β, but also for the drift parameter α and the volatility parameter σ; and (iii) empirically, the proposed estimation procedure is applied to check whether the LRD or SRD property of two well–known financial data sets: a) the S&P 500 Stock Price Index and b) the Treasury Bill rate. In addition, the data simulation procedure proposed in this paper is worthy of note. A set of values of the form (5) are generated. Comte and Renault (1998) generate values using form (4), however, the stationarity of the random vector $X(t)$ is not assured. Once the auto–covariance function is known, the auto–covariance matrix can be generated as a symmetric matrix. This property can be used to generate the required Gaussian random vector from a standard Gaussian random vector. The main advantage of this method is its computational efficiency and simplicity. The time spent in the generation of a random vector depends on the length of the vector. On the other hand, a numerical approximation to solution (5) depends on the vector length and the chosen step size. Often a small step size is needed to obtain a good convergence of the numerical approximation which makes the numerical approach slow and even impractical.

This paper is organised as follows. Section 2 proposes the estimation procedure and then establishes the corresponding asymptotic theory. The numerical implementation of the proposed estimation procedure is described in Section 2.2. A theoretical explanation of the way this numerical procedure may be applied to the FSV process completes Section 2. Results of the simulations that were carried out can be found in Section 3. The estimation procedure is tested with simulated data and compared to the model proposed by Comte and Renault (1998) to solve the FSV process. Then, the estimation procedure is applied to real financial data. Our results support those obtained by Ding, Granger and Engle (1993). Mathematical details are relegated to Appendix A. Appendix B contains all the tables and figures. This paper finishes with a summary of the results.

2. ESTIMATION PROCEDURE

2.1 Continuous–time Estimation Procedure

The spectral density function $\phi(\omega, \theta)$ given in equation (3) is well–defined for all values $\omega \in \Re$. Thus for values of $\beta \in (0, \frac{1}{2})$, the spectral density behaves as a usual LRD spectral density: decreasing to zero as $|\omega| \to \infty$ and increasing to ∞ as $|\omega| \to 0$. For values of $\beta \in (-\frac{1}{2}, 0)$ the spectral density, $\phi(\omega, \theta)$, decreases to zero as $|\omega| \to \infty$ and $|\omega| \to 0$ and has the maximum at $\omega = \alpha\sqrt{\frac{-\beta}{1+\beta}}$.

Some detailed discussion about spectral analysis involving short–lange dependent stationary time series can be found in §10 of Brockwell and Davis (1991) and Priestly (1981). For the LRD case, Gao, Anh and Heyde (2002) propose a continuous–time periodogram of the form

$$I_N^Y(\omega) = \frac{1}{2\pi N}\left|\int_0^N e^{-i\omega t} Y(t)dt\right|^2, \tag{8}$$

where $N > 0$ is the upper bound of the interval $[0, N]$, on which each $Y(t)$ is observed.

As in Gao (2004), this paper uses an extended continuous–time version of the discrete Gauss–Whittle contrast function used by Dahlahaus (1989) of the form

$$L_N^Y(\theta) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left\{\log(\phi(\omega,\theta)) + \frac{I_N^X(\omega)}{\phi(\omega,\theta)}\right\}\frac{d\omega}{1+\omega^2}. \tag{9}$$

We then define the minimum contrast estimtor of θ as

$$\bar{\theta}_N = \arg\min_{\theta\in\Theta_0} L_N^Y(\theta), \tag{10}$$

where Θ_0 is a compact subset of Θ.

As can be seen from Theorem 3.1 of Gao (2004), both the convergence in probability and the asymptotic normality of $\bar{\theta}_N$ hold automatically for the case where $\theta \in \Theta_1 = \left\{\theta = (\alpha,\beta,\sigma) : \alpha > 0, 0 < \beta < \frac{1}{2}, \sigma > 0\right\}$. The following theorem shows that such consistency results also hold for the general case where $\theta \in \Theta$.

Theorem 1 *(i) Let θ_0 be the true value of θ. Then, as $N \to \infty$*

$$P\left(\lim_{N\to\infty}\bar{\theta}_N = \theta_0\right) = 1.$$

(ii) In addition, if the true value θ_0 of θ is in the interior of Θ_0, then, as $N \to \infty$

$$\sqrt{N}(\bar{\theta}_N - \theta_0) \to_D N\left(0, \Sigma^{-1}(\theta_0)\right),$$

where

$$\Sigma(\theta) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\frac{\partial\log(\phi(\omega,\theta))}{\partial\theta}\right)\left(\frac{\partial\log(\phi(\omega,\theta))}{\partial\theta}\right)^{\tau}\frac{1}{(1+\omega^2)^2}d\omega,$$

in which $\frac{\partial\log(\phi(\omega,\theta))}{\partial\theta} = \left(-\frac{2\alpha}{\omega^2+\alpha^2}, -2\frac{\Gamma'(1+\beta)}{\Gamma(1+\beta)} - \log(w^2), \frac{2}{\sigma}\right)^{\tau}$.

The proof of the theorem is relegated to Appendix A below. In the following section, we discuss how to realize the proposed continuous–time estimation procedure as well as to illustrate the asymptotic consistency results in practice.

2.2 Discrete Estimation Procedure

In many practical circumstances, observations on $Y(t)$ are made at discrete intervals of time, even though the underlying process may be continuous. In addition, it is computationally easier to find a consistent estimate of θ based on a sequence of discrete observations on $Y(t)$. This section considers the following discrete process:

$$Z_t = Y(t), \quad t = 1, 2, \ldots, T-1 \quad \text{and } T = [N].$$

Such $\{Z_t\}$ is stationary and normally distributed with $\mathbb{E}[Z_t] = 0$ and auto-covariance function obtained as the inverse Fourier transform of its density function. As can be seen in equation (2), $\phi(\omega, \theta)$ is symmetric with respect to ω and therefore the complex terms of the transformation cancel out. Thus, the auto–covariance function is calculated as follows:

$$\gamma(\tau) = 2 \int_0^\infty \phi(\omega, \theta) \cos(\omega\tau) d\omega. \tag{11}$$

Equivalently, $\phi(\omega, \theta)$ is the Fourier transform of the covariance function of the stationary process $\{Z_t\}$ (see Priestly 1981) given by

$$\phi(\omega, \theta) = \frac{1}{2\pi} \int_{-\infty}^\infty \gamma(\tau) e^{-i\tau\omega} d\omega.$$

It can be seen from Bloomfield (1976, §2.5) that the corresponding spectral density of $\{Z_t\}$ is defined by

$$f_Z(\omega) = f(\omega, \theta) = \sum_{k=-\infty}^{\infty} \phi(\omega - 2k\pi, \theta). \tag{12}$$

Given T observations $Z_1, \ldots, Z_T$, we may estimate the spectral density function $f_Z(w, \theta_0)$ by

$$I_T^Z(\omega) = \frac{1}{2\pi T} \left| \sum_{t=1}^T e^{-i\omega t} Z_t \right|^2. \tag{13}$$

As a discrete approximation to $L_N^Y(\theta)$ defined in Section 2, we use a discrete version of the form

$$W(\theta) = W_T(\theta) = \frac{1}{2T} \sum_{s=1}^{T-1} \left\{ \log(f(w_s, \theta)) + \frac{I_T^Z(w_s)}{f(w_s, \theta)} \right\} \tag{14}$$

with $w_s = \frac{2\pi s}{T}$.

Thus, the (discrete) minimum contrast estimator of θ can be defined by

$$\hat{\theta}_T = \arg\min_{\theta \in \Theta_0} W(\theta), \tag{15}$$

which approximates $\bar{\theta}_N$ for large enough N, i.e. it can be shown that

$$\lim_{N\to\infty} P(|\hat{\theta}_T - \bar{\theta}_N| \geq \epsilon) = 0$$

for any given small $\epsilon > 0$. Thus, we may approximate $\bar{\theta}_N$ by $\hat{\theta}_T$ in Section 3 below.

3. SIMULATIONS AND APPLICATIONS

3.1 Data Simulation

To validate the results of the estimation procedure, sample paths with predetermined parameters and distribution are generated. The aim of this simulation is to estimate these parameters as accurately as possible to prove that the estimation procedure is reliable in practice.

The solution described by equation (5) is stationary and normally distributed. A discrete sample path, $\{Z_t : t = 1, \ldots, T\}$, is generated in the following manner:

- generate C_T, a $T \times T$ auto-covariance matrix, using the auto-covariance function given by equation (11) with $\tau = 1$. C_T is a symmetric non-negative definite matrix with spectral decomposition $C_T = V \Lambda V^\top$, where $\Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_T\}$ is the diagonal matrix of the eigenvalues and V is the orthogonal matrix of the eigenvectors such that $V^\top V = I$ with $V^\top$ being the matrix transpose of V;
- generate a sample $G = (g_1, g_2, \ldots, g_T)^\top$ of independent realisations of a multivariate Gaussian random vector with the zero vector as the mean and the identity matrix as the covariance matrix; and
- generate $(z_1, \ldots, z_T) = V \Lambda^{1/2} V^\top G$ as the realisation of a multivariate Gaussian random vector with the zero vector as the mean and C_T as the covariance matrix.

The sample path generated with the initial parameter values $\theta_0 = (\alpha, \beta, \sigma) = (1.5, 0.1, 0.01)$ is illustrated in Figure 1. The periodogram and the spectral density of the simulated data set are illustrated in Figure 2.

Figures 1 and 2 near here

3.2 Estimation of θ

Samples for different parameters, θ_0, and different lengths, T, were generated. The discrete estimation procedure explained in Section 2.2 was applied to these samples to obtain estimators of θ_0. The aim was to show that the estimation procedure was reliable for any sample path that follows a model of the form (2).

Tables 1, 2, 3 and 4 near here

The results in Tables 1, 2, 3 and 4 show the empirical means, the empirical standard deviations and the empirical mean square errors. When the number of points generated increases the empirical mean gets closer to the value of the real parameter and its standard error reduces. This shows that there is an asymptotic convergence of the estimates to the real parameters. The estimates obtained with 100 and 1000 simulations do not differ strongly from each other. This may show that the method is also robust for small numbers of simulations. The tables show that the parameter β may be estimated quite accurately. In addition, the simulations have been carried out for values of $\beta \in (-\frac{1}{2}, \frac{1}{2})$ including the case with $\beta = 0$ that is shown separately in Table 5. The results confirm that this procedure can be used to estimate the parameters of financial data with possible LRD or SRD.

Table 5

The parameter β is restricted to the interval $(-\frac{1}{2}, \frac{1}{2})$, whereas α and σ can take any positive value. Large values of α and σ need larger data sets for some good estimation. For instance in the simulation for $\theta = (1, -0.2, 0.05)$, there were 15 outliers out of 1000 estimates. This is less than 2% of the number of estimates but these have a large effect on the empirical mean. As the size of the data increases, the occurrence of these outliers decreases. Table 6 displays the estimates for two parameters with large α and σ values.

Table 6

In summary, the proposed estimation procedure works well numerically.

3.3 Estimation of β for the FSV Model with Two Procedures

Given a financial time series, one of the questions that arises is whether the data exhibit LRD. The most common way to quantify LRD is by assessing the Hurst index and in particular, estimating the β parameter. The comparison of two procedures to estimate β involved in the FSV model (7) is discussed in this Section. *Procedure A* refers to the estimation procedure described in Section 2.2 and *procedure B* refers to the estimation procedure presented by Comte and Renault (1998).

One hundred discrete sample paths, $\{Z_t\}$, of length 400 with parameter $\theta_0 = (3, 0.3, 1)$ are generated as described in Section 3.1. Two simulation approaches are compared: i) the estimation of β involved in the stochastic logarithm volatility process, $\ln(v(t))$, where the estimation procedure is applied over the sample path $\{Z_t\}$ and ii) the estimation of β involved in the stochastic volatility process, $v(t)$, where the proposed estimation procedure is applied over the sample path $\{\exp(Z_t)\}$. The results with procedure A and procedure B (results for procedure B are taken from Section 6.4 of Comte and Renault 1998) are displayed in Table 7.

Table 7 near here

The main difference between the two procedures is the way in which the data is simulated. Procedure B generates discrete sample paths of the process (4) (see Section 3.3 of Comte and Renault 1998). The convergence of this procedure as well as the speed depends on the step size, h. Procedure A generates stationary sample paths as Gaussian random vectors with mean vector zero and auto–covariance function (11). The other important difference between the two procedures is that procedure B uses the log-periodogram as the discrete density estimate of Z_t while procedure A uses the periodogram as introduced above.

It can be seen in Table 7 that for sets of 400 records procedure A performs better than procedure B for simulations of the process $\ln(v(t))$. The standard deviation of procedure A decreases when the size of the sample increases as can be seen from Table 8, which shows the estimators of β improves drastically without a very high computational cost.

Table 8 near here

In summary, procedure A is a good choice to estimate β for the FSV model (7). It obtains good results and is computationally cheap. The latter is discussed in more detail in the following subsection.

3.4 Computational Performance Comparison

The essential advantage of the data simulation with procedure A for the estimation of β is its computational efficiency. The generation of 100 samples of length 400 of the process $v(t)$ with procedure A takes approximately half an hour (CPU user time). However, the generation of 100 samples of length 400 using procedure B and step size $h = 1/20$ has been estimated to take more than two days (CPU user time) which makes procedure B unfeasible for large sample paths. Although computer power nowadays has improved greatly, this task can make the estimation algorithm very slow and impractical. Therefore a feasible and efficient numerical approach is desirable.

Simulations to compare the performance between the two procedures were carried out on a Pentium 4 (2.4 GHz), using the R programming language. Sample paths, $\{Z_t : t = 1, \ldots, T\}$ for different lengths, T, were generated with procedure A and procedure B. The performance of the generation process of both procedures are shown in Table 9 and Figure 3. These show the CPU user time as a function of the number of points generated.

Table 9 and Figure 3 near here

The efficiency of procedure A permits large sample paths to be generated to ensure a good parameter estimation.

3.5 Financial Data

A good estimation procedure must be able to solve some real data problems if it is to be of any practical value. To test whether the proposed estimation procedure works adequately for real data, two data sets have been studied:

i) the daily values of the S&P 500 Stock Price Index from January 1928 to December 1987 and,

ii) the monthly values of the three–month Treasury Bill rate from January 1963 to December 1998.

The first step is to prepare the data under study such that a set of stationary Gaussian data can be obtained. In this Section, two transformations to produce stationary data are considered:

- The first difference of the original data set is defined as follows

$$V_t = Z_t - Z_{t-1} \text{ for } t = 1 \ldots T. \tag{16}$$

- The compounded return of $\{Z_t\}$ is the first difference of the natural logarithm of the original data set, given by

$$W_t = \ln\left(\frac{Z_t}{Z_{t-1}}\right) \text{ for } t = 2 \ldots T. \tag{17}$$

In some of the cases studied in this paper, once the stationarity was assured, the data needed to be slightly truncated to ensure Gaussianity.

3.5.1 S&P 500

For the first financial example, a section of the S&P 500 Stock Price Index from January 1928 to December 1987 is considered and four subsets taken: the whole set with 16,128 daily values; a set of 10,000 daily values from the 21^{st} of September 1948; a set of 2,000 daily values from the 4^{th} of February 1980; and a set of 500 daily values from the 10^{th} of January 1986. The trajectory of the S&P 500 Stock Price Index is illustrated in Figure 4.

Figure 4 near here

The initial data set, $X(t)$, is transformed to obtain a stationary set using equation (17). Afterwards, the new data set is truncated by the 1% and 99% quantiles to assure normality.

Next, equations (12) to (14) are applied to the transformed set. The estimates of the parameters involved in the density function (3) of the S&P 500 Stock Price Index are found. These are shown in Table 10. The estimate of the spectral density is shown in Figure 5.

Table 10 and Figure 5 near here

The β estimates that correspond to the two large data sets within the interval $(0, \frac{1}{2})$ suggest that these sets may display LRD. However, the β estimates corresponding to the two smaller sections of the data set are negative therefore, the smaller sets do not display LRD. For the two larger sections of the data, moreover, our findings are consistent with those results obtained by Ding, *et al.* (1993). They analyse the autocorrelation function (ACF) of the compounded return, W_t, $|W_t|$ and W_t^2 for a large section of the S&P 500 Stock Price Index, from January 1928 to August 1991. Their analysis is repeated in this paper for different sections of the S&P 500 Stock price Index compounded return. The results are shown in Figures 6 and 7 and Table 11. In these figures, the ACF dies off for the smaller sets, but it is still important for large lags for the larger sets.

Table 11 near here

Figures 6 and 7 near here

In summary, our studies show that there is some weak evidence of LRD for large sections of the S&P 500 Stock Index Price, while small sections of the data exhibit SRD.

3.5.2 T–Bill rate

The T–Bill rate, shown in Figure 8, are monthly observations over the period from January 1963 to December 1998. An initial look at the data suggests that this set does not exhibit stationarity. This can be achieved with the appropriate transformation. The two transformations described by equations (16) and (17) were applied to this data. In some cases truncations were needed to ensure Gaussianity.

Figure 8 near here

The parameters α, β and σ are then estimated applying equations (12) to (14) as before. The resulting estimates are shown in Table 12. The density function estimate is shown in Figure 9. For each of the transformations, the estimate of β is negative and therefore does not suggest that the data set may display LRD.

Table 12 and Figure 9 near here

The ACF of V_t and W_t, as well as their absolute values and the square values of the transformed data are examined. The functions are displayed in Figure 10 for the first difference and in Figure 11 for the compounded return. As can be seen from Table 13, the autocorrelation values die off for long lags, i.e. the data does not display LRD as was acknowledged by the results obtained with the estimation procedure discussed above.

Figures 10 and 11 near here

Table 13 near here

4. Discussion

Recently, several methods and models have been proposed to model data with LRD property. This paper has extended one of the models proposed in Gao (2004) to accommodate cases where some sections of the data may exhibit LRD while other sections may not exhibit LRD. Such an extension has then been applied to examine both the S&P 500 index and the T-Bill rate. For the the S&P 500 index, our studies have indicated that there is some kind of weak evidence of LRD for the data values recorded before 1950. In addition, our research provides a kind of answer to the question of whether or not the T–Bill rate should be treated as long–range dependent time series. We conclude that the T-Bill rate does not exhibit LRD.

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A. Proof of Theorem 1

In order to prove Theorem 1, we need to introduce the following assumption and then a lemma.

Assumption A.1 (i) Assume that $\phi(\omega, \theta)$ satisfies

$$\int_{-\infty}^{\infty} \phi(\omega, \theta)\ d\omega < \infty$$

for all $\theta \in \Theta = \left\{0 < \alpha < \infty, -\frac{1}{2} < \beta < \frac{1}{2}, 0 < \sigma < \infty\right\}$.

(ii) Let θ_0 be the true value of θ, and θ_0 be in the interior of Θ_0, a compact subset of Θ. For any small $\epsilon > 0$, if $\epsilon < ||\theta - \theta_0|| < \frac{1}{4}$ then

$$\int_{-\infty}^{\infty} \frac{\phi(\omega, \theta_0)}{\phi(\omega, \theta)} \frac{1}{1+\omega^2}\ d\omega < \infty,$$

where $||\cdot||$ denotes the Euclidean norm.

(iii) For any real function $h(\cdot) \in L^2(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} \frac{h^2(\omega, \theta_0)}{(1+\omega^2)^2} \left(\frac{\partial \log(\phi(\omega, \theta))}{\partial \theta}\right)^{\tau} \left(\frac{\partial \log(\phi(\omega, \theta))}{\partial \theta}\right) |_{\theta=\theta_0}\ d\omega < \infty,$$

where $\frac{\partial \log(\phi(\omega,\theta))}{\partial \theta} = \left(\frac{\partial \log(\phi(\omega,\theta))}{\partial \alpha}, \frac{\partial \log(\phi(\omega,\theta))}{\partial \beta}, \frac{\partial \log(\phi(\omega,\theta))}{\partial \sigma}\right)^{\tau}$.

(iv) For $\theta \in \Theta$,

$$\Sigma(\theta) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\frac{\partial \log(\phi(\omega,\theta))}{\partial\theta}\right)\left(\frac{\partial \log(\phi(\omega,\theta))}{\partial\theta}\right)^{\tau}\frac{1}{(1+\omega^2)^2}\,d\omega < \infty.$$

(v) The inverse matrix, $\Sigma^{-1}(\theta_0)$, of $\Sigma(\theta_0)$ exists.

Assumption A.1 is a set of modified versions of Conditions 2.1–2.3 of Gao, Anh and Heyde (2002). In order to ensure that possible cases of short–range dependence can be included, the parameter space Θ has been expanded to cover the case where $-\frac{1}{2} < \beta \leq 0$. Under Assumption A.1, we have the following lemma.

Lemma A.1. Assume that Assumption A.1 holds. Let θ_0 be the true value of θ. Then, as $N \to \infty$

$$P\left(\lim_{N\to\infty}\bar{\theta}_N = \theta_0\right) = 1.$$

In addition, if the true value θ_0 of θ is in the interior of Θ_0, then, as $N \to \infty$

$$\sqrt{N}(\bar{\theta}_N - \theta_0) \to_D N\left(0, \Sigma^{-1}(\theta_0)\right),$$

where $\Sigma(\theta)$ is as defined in Assumption A.1(iv).

The proof of Lemma A.1 is similar to that of Theorem 2.4 of Gao, Anh and Heyde (2002). As may be seen from the detailed proof of the Theorem 2.4, all the derivations remain true under Assumption A.1.

Proof of Theorem 1. It follows from Lemma A.1 that in order to prove Theorem 1, it suffices to verify Assumption A.1 in detail.

To verify Assumption A.1, we need only to consider the case where

$$\theta \in \Theta_2 = \left\{\alpha > 0, -\frac{1}{2} < \beta \leq 0, \sigma > 0\right\},$$

since the verification for the case where $\theta \in \Theta_1 = \left\{\alpha > 0, 0 < \beta < \frac{1}{2}, \sigma > 0\right\}$ can be done similarly.

For the case where $-\frac{1}{2} < \beta \leq 0$, let $\gamma = -\beta$. Then $0 \leq \gamma < \frac{1}{2}$. We now rewrite the spectral density function $\phi(\omega, \theta)$ as

$$\phi(\omega) = \phi(\omega,\theta) = \frac{\sigma^2}{\Gamma^2(1-\gamma)}\frac{|\omega|^{2\gamma}}{\omega^2+\alpha^2}, \quad \omega \in (-\infty, \infty). \tag{18}$$

Thus, for $0 < \gamma < \frac{1}{2}$ we have

$$\int_{-\infty}^{\infty}\phi(\omega,\theta)d\omega = \int_{-\infty}^{\infty}\frac{\sigma^2}{\Gamma^2(1-\gamma)}\frac{|\omega|^{2\gamma}}{\omega^2+\alpha^2}\,d\omega = \frac{\sigma^2}{\Gamma^2(1-\gamma)}\int_{-\infty}^{\infty}\frac{|\omega|^{2\gamma}}{\omega^2+\alpha^2}\,d\omega < \infty \tag{19}$$

using $\frac{\omega^{2\gamma}}{\omega^2+\alpha^2} \approx |\omega|^{2\gamma}$ as $\omega \to 0$ with $\int_{c_1}^{c_2} |\omega|^{2\gamma}\, d\omega < \infty$ for any given small $c_1 < c_2$, and $\frac{\omega^{2\gamma}}{\omega^2+\alpha^2} \approx \frac{1}{\omega^{2(1-\gamma)}}$ as $\omega \to \infty$ with $\int_C^{\infty} \frac{1}{\omega^{2(1-\gamma)}}\, d\omega < \infty$ for any given $C > 0$. This implies that Assumption A.1(i) holds.

Similarly, we may verify that Assumption A.1(ii) also holds using the fact that $|\beta - \beta_0| = |\gamma - \gamma_0|$ when $\gamma = -\beta$ and $\gamma_0 = -\beta_0$ are used.

Assumption A.1(iii) follows from the assumption that $h(\cdot) \in L^2(-\infty, \infty)$ and

$$\left(\frac{\partial \log(\phi(\omega,\theta))}{\partial \theta}\right)^{\tau} \left(\frac{\partial \log(\phi(\omega,\theta))}{\partial \theta}\right) |_{\theta=\theta_0}$$

$$= \frac{4}{\sigma_0^2} + \frac{4\alpha_0^2}{(\omega^2+\alpha_0^2)^2} + \left(2\frac{\Gamma'(1-\gamma_0)}{\Gamma(1-\gamma_0)} + \log(w^2)\right)^2$$

using

$$\frac{\partial \log(\phi(\omega,\theta))}{\partial \theta} = \left(-\frac{2\alpha}{\omega^2+\alpha^2}, -2\frac{\Gamma'(1+\beta)}{\Gamma(1+\beta)} - \log(w^2), \frac{2}{\sigma}\right)^{\tau}. \tag{20}$$

The verification of Assumption A.1(iv)(v) can be done similarly using (20). We thus finish the verification of Assumption A.1. The proof of Theorem 1 is therefore finished.

B. Figures and Tables

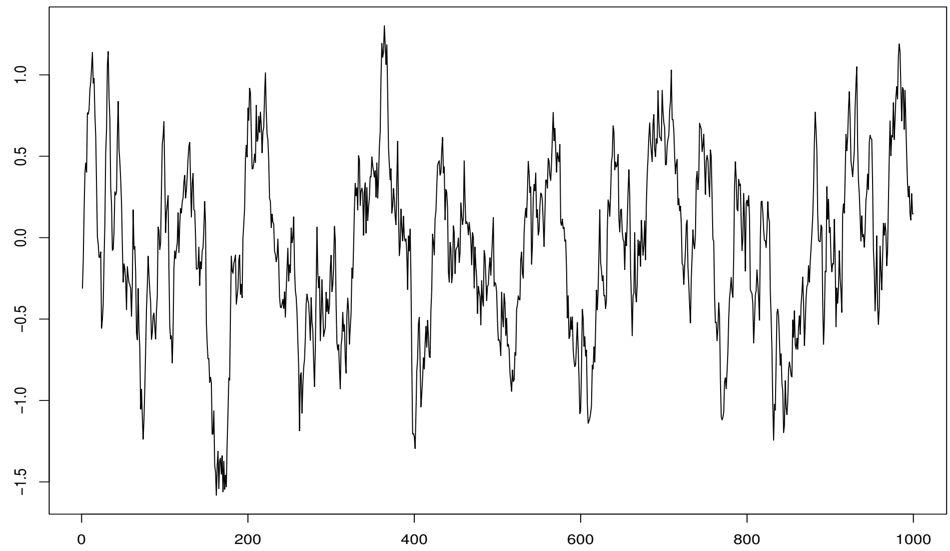


Figure 1: Sample path for data generated with $\theta = (\alpha, \beta, \sigma) = (0.1, 0.1, 0.1)$ and $\tau = 1$.

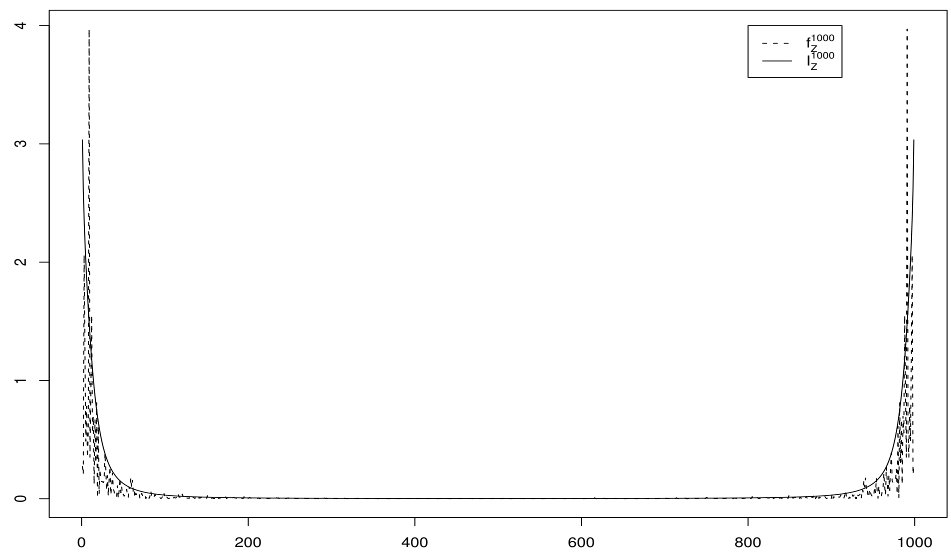


Figure 2: The periodogram and the spectral density for $\theta = (\alpha, \beta, \sigma) = (0.1, 0.1, 0.1)$ and $\tau = 1$.

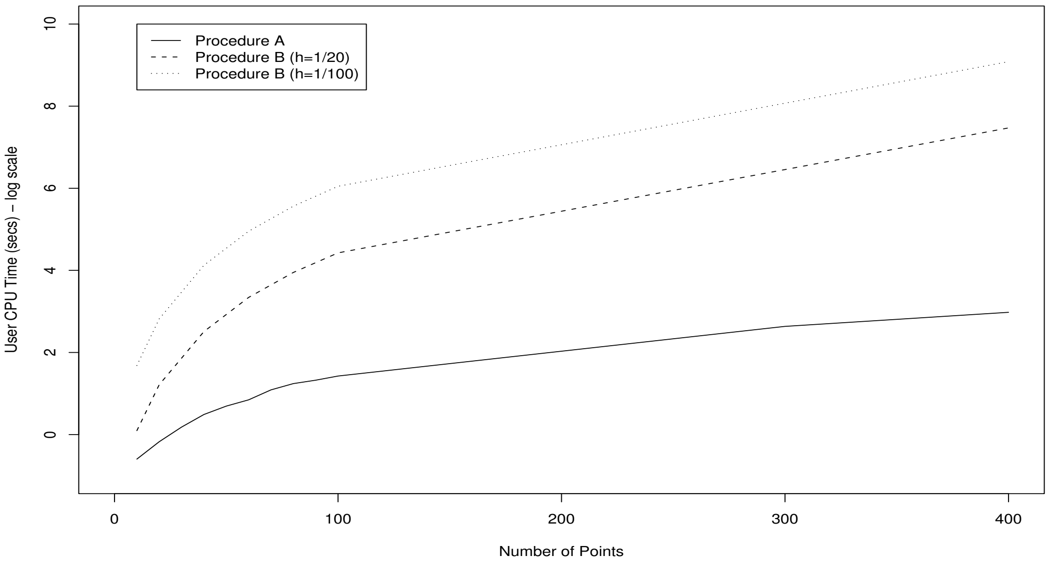


Figure 3: Performance comparison between the data set generation of procedure A and procedure B.

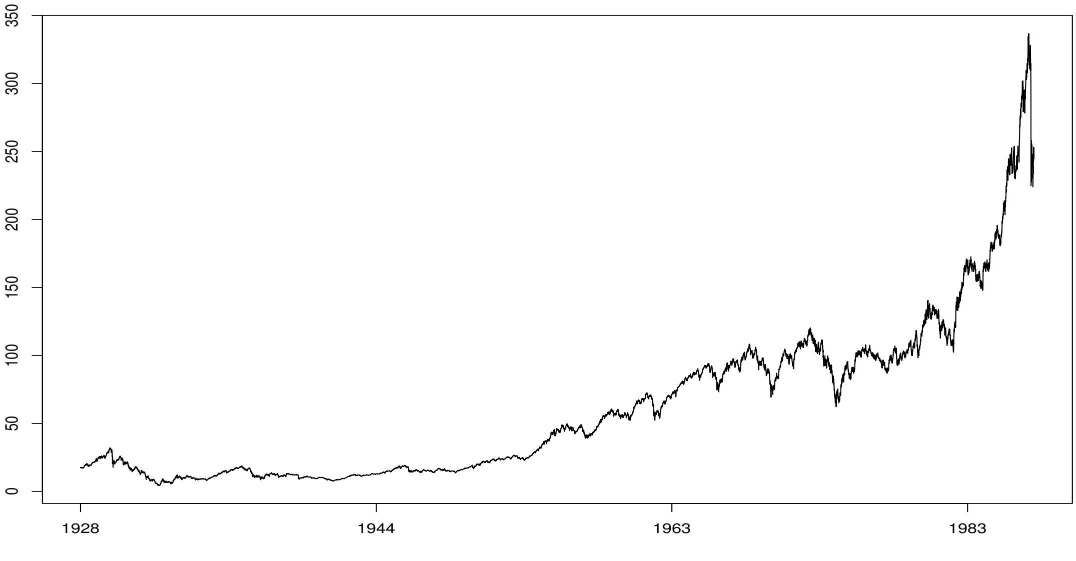
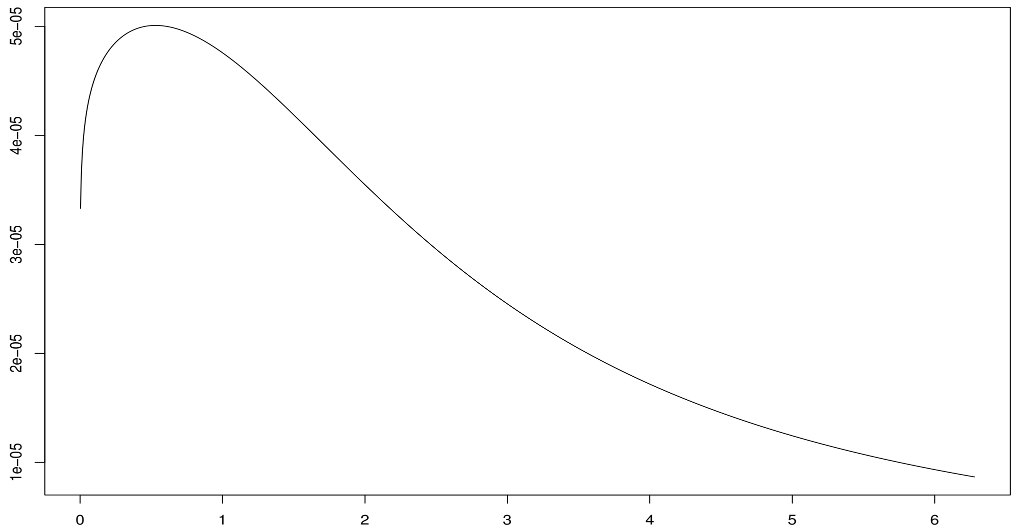
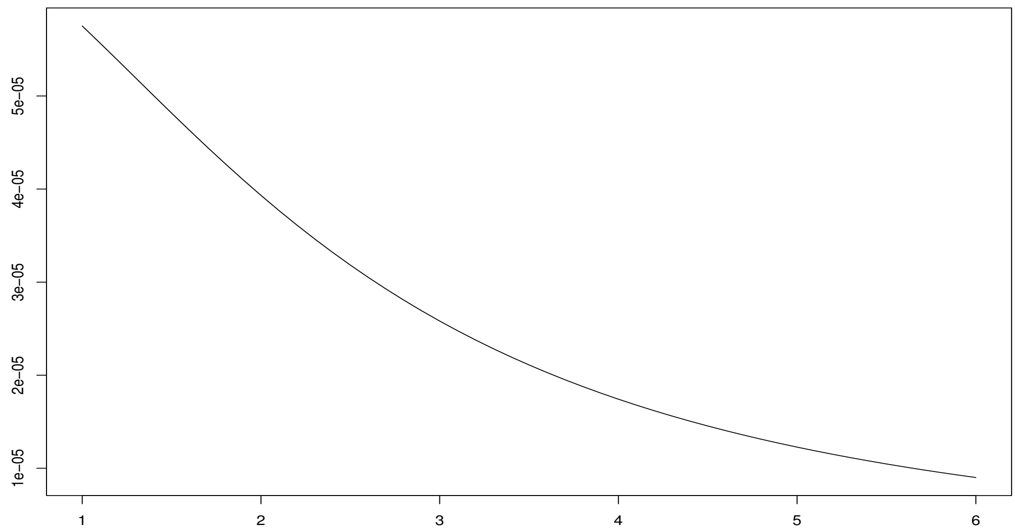


Figure 4: A Section of the S&P 500 Index from Jan. 1928 to Dec. 1987.

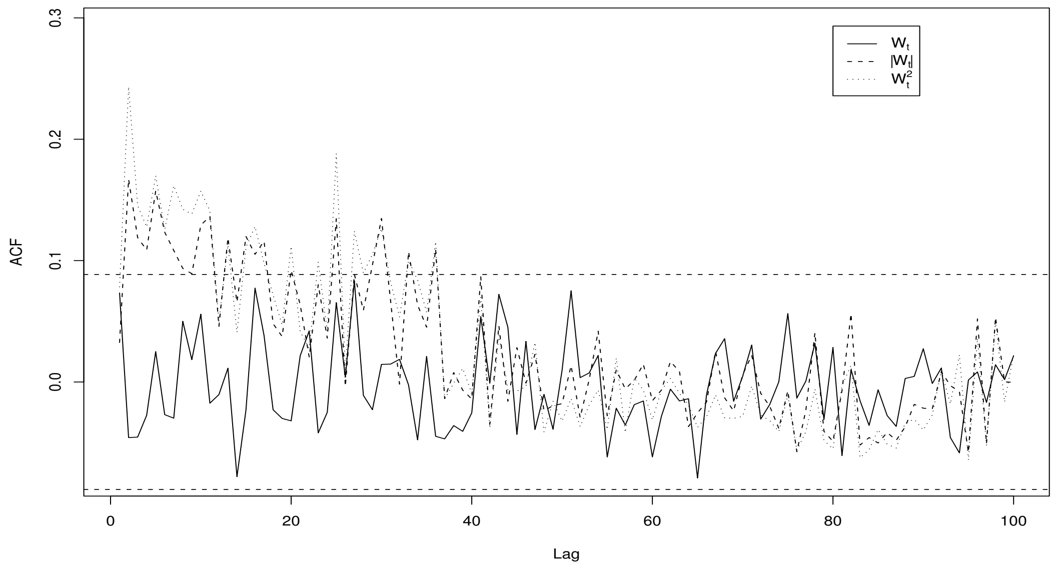


a)

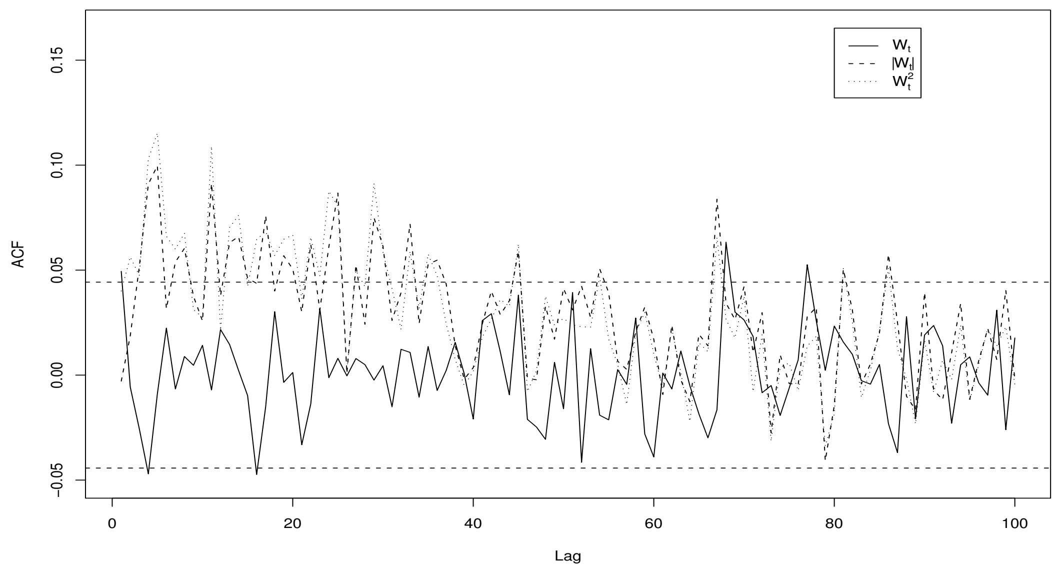


b)

Figure 5: Estimate of the spectral density function referring to the truncated compounded return of sections of the S&P 500 Index: a) $(\hat{\alpha}, \hat{\beta}, \hat{\sigma})$ = (2.4566, -0.0444, 0.0188), b) $(\hat{\alpha}, \hat{\beta}, \hat{\sigma})$ = (2.4165, 0.0120, 0.0197).

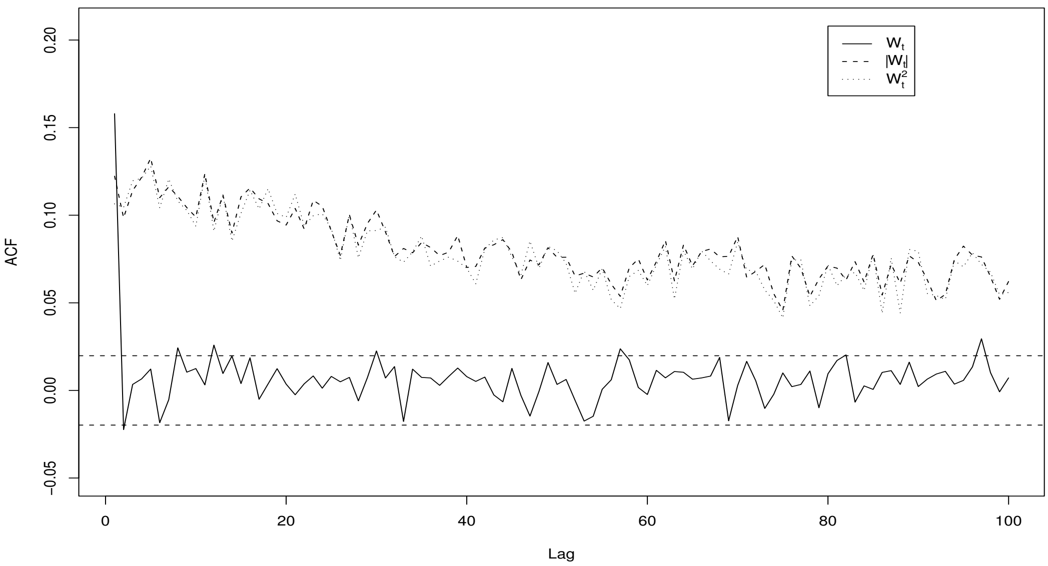


a)

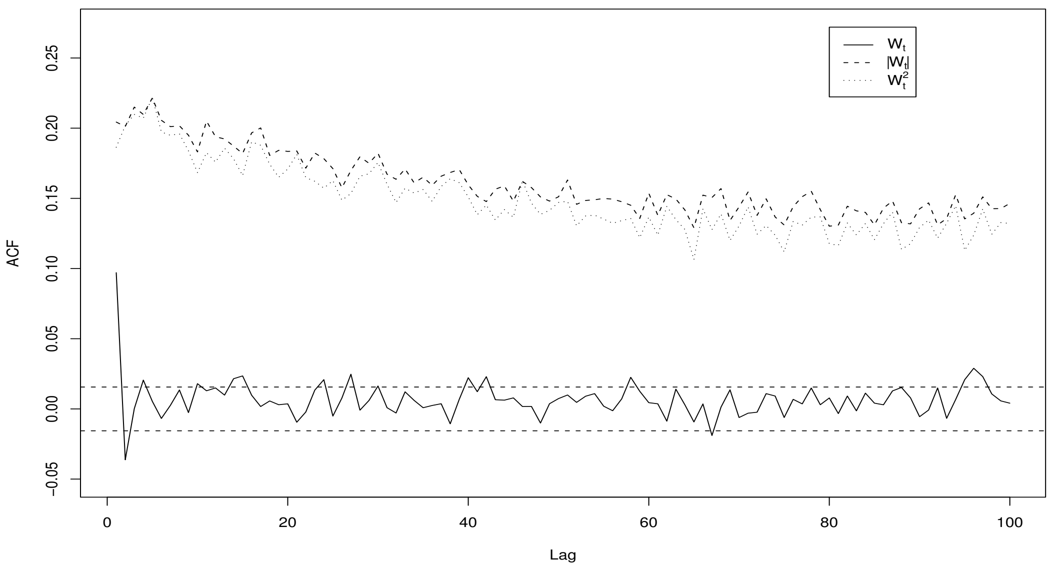


b)

Figure 6: ACF for W_t, $|W_t|$ and W_t^2 of the: a) S&P 500 referring to Jan. 1986 to Dec. 1987, b) S&P 500 referring to Feb. 1980 to Dec. 1987.



a)



b)

Figure 7: ACF for W_t, $|W_t|$ and W_t^2 of the: a) S&P 500 referring to Sep. 1948 to Dec. 1987, b) S&P 500 referring to Jan. 1928 to Dec. 1987.

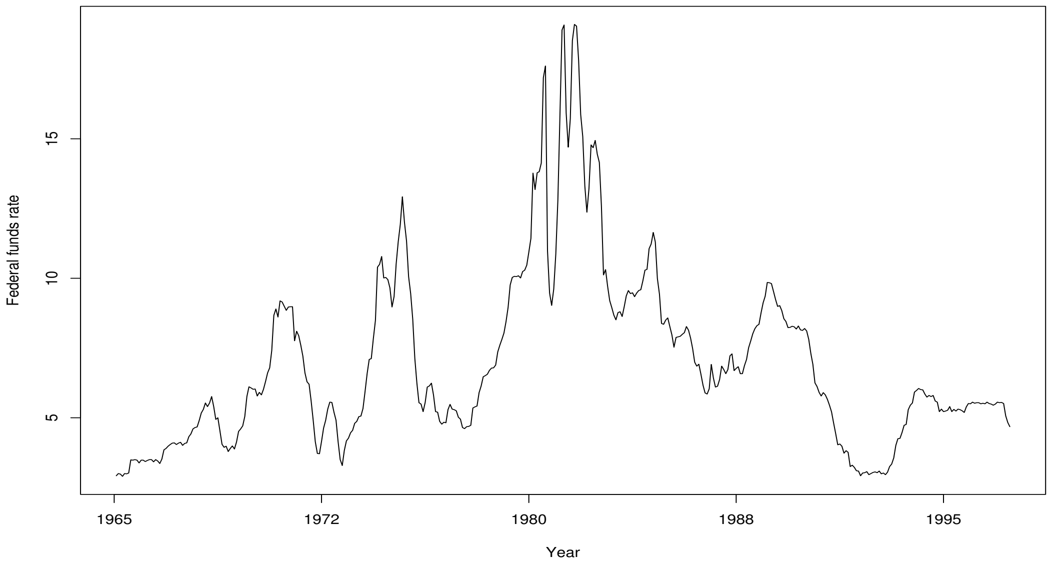


Figure 8: Original T-Bill rate.

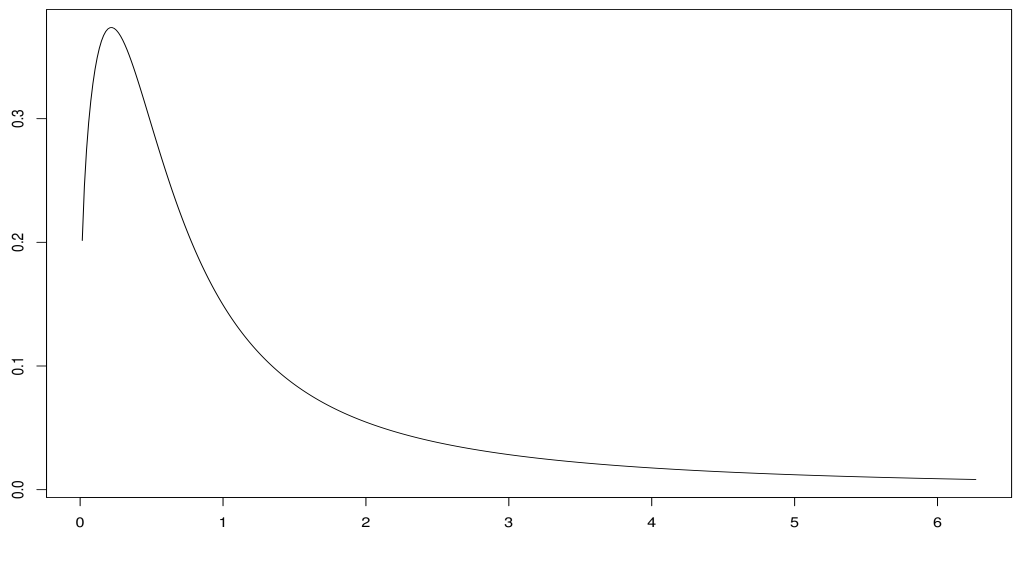
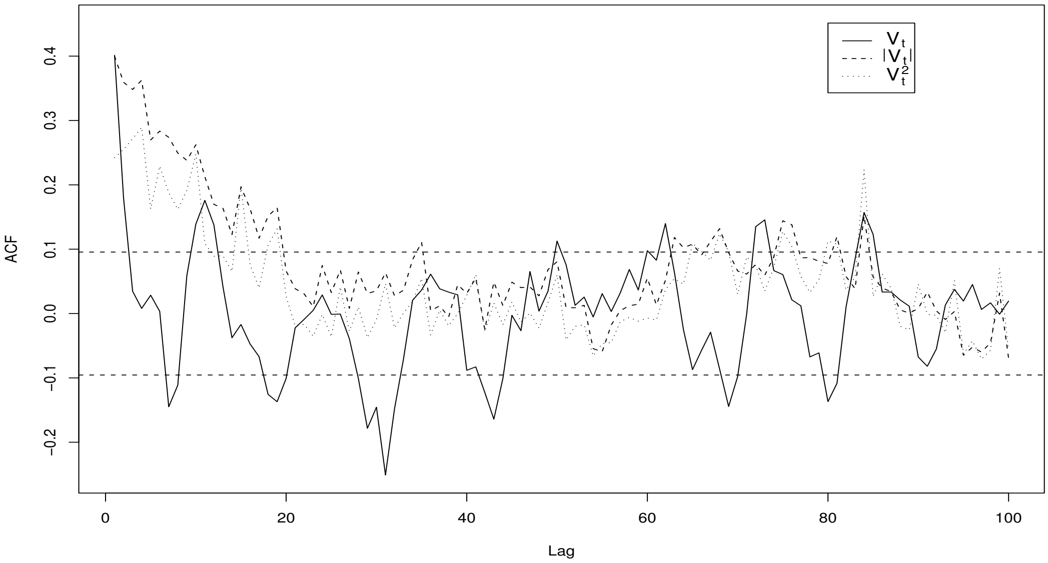
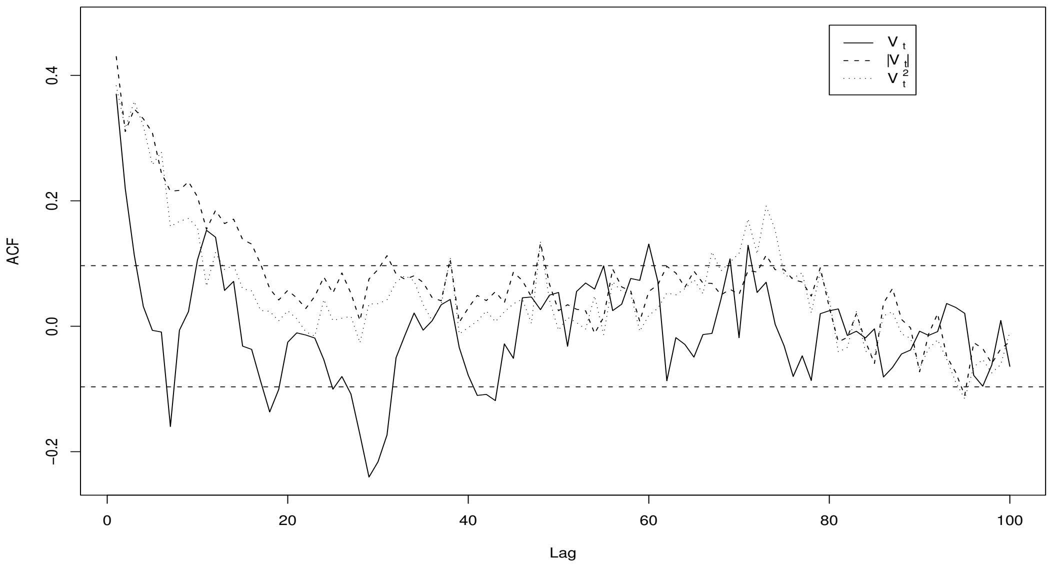


Figure 9: Estimate of the spectral density function with $(\hat{\alpha}, \hat{\beta}, \hat{\sigma}) = (0.5322, -0.1440, 0.4846)$.

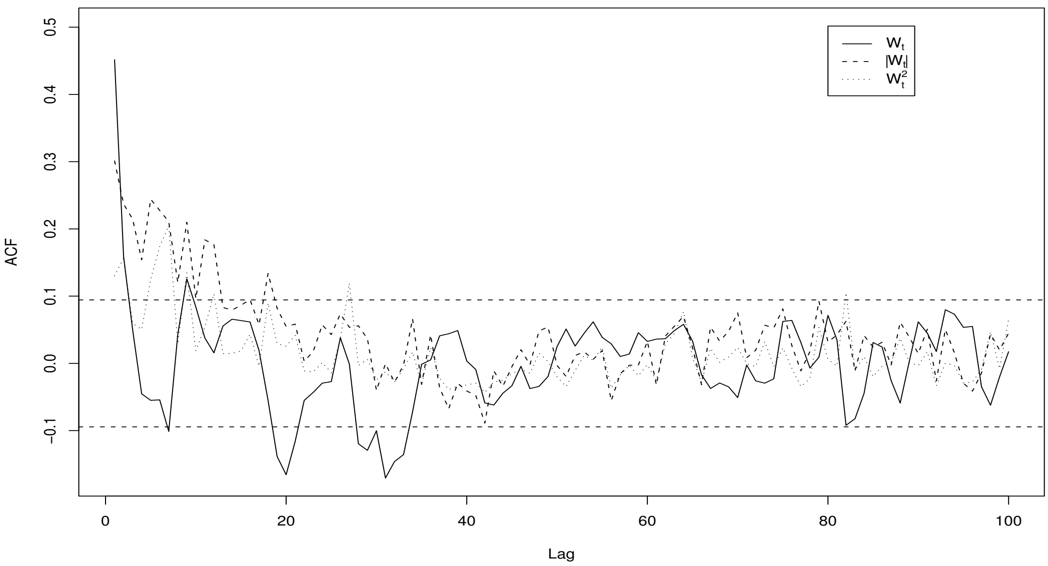


a)

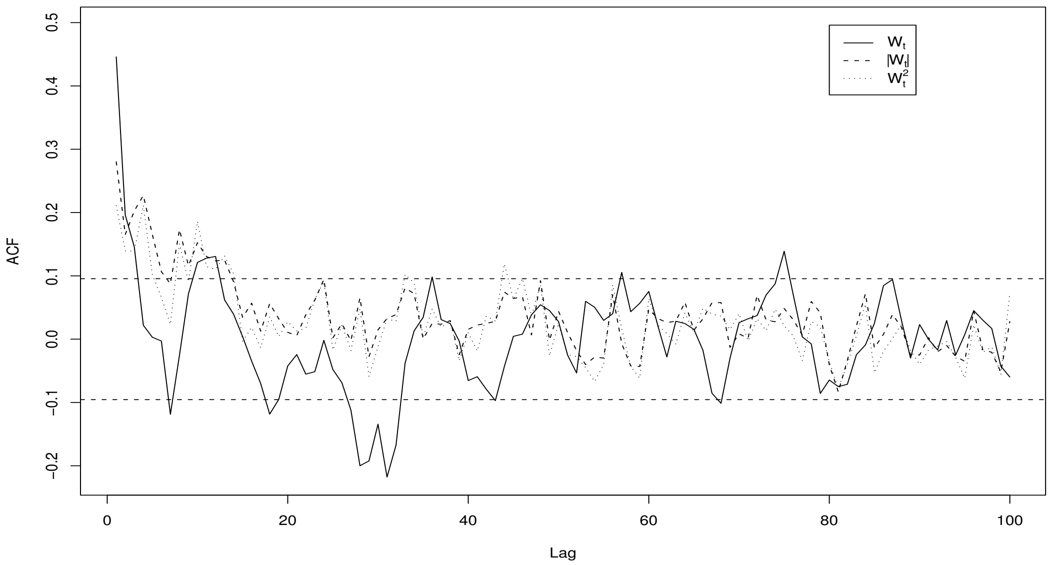


b)

Figure 10: ACF for V_t, $|V_t|$ and V_t^2 of the first difference of the T-Bill rate: a) truncated by the 1% and 99% quantiles, b) truncated by the 2% and 98% quantiles.



a)



b)

Figure 11: ACF for W_t, $|W_t|$ and W_t^2 of the compounded return of the T-Bill rate: a) without truncation, b) truncated by the 1% and 99% quantiles.

	$T = 400$			$T = 1000$			$T = 2500$		
$\theta_0 = (0.1, 0.1, 0.1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1120	0.1015	0.0731	0.1078	0.1036	0.0714	0.1038	0.1012	0.0713
Empirical std.dev.	0.0484	0.0620	0.0042	0.0257	0.0329	0.0021	0.0168	0.0218	0.0011
Empirical MSE	0.0025	0.0038	0.0007	0.0007	0.0011	0.0008	0.0003	0.0005	0.0008
$\theta_0 = (0.1, -0.1, 0.1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1144	-0.0939	0.0715	0.1052	-0.0984	0.0711	0.1003	-0.1035	0.0710
Empirical std.dev.	0.0488	0.0457	0.0029	0.0264	0.0295	0.0017	0.0170	0.0196	0.0010
Empirical MSE	0.0026	0.0021	0.0008	0.0007	0.0009	0.0008	0.0003	0.0004	0.0008
$\theta_0 = (0.1, 0.2, 0.01)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1145	0.1958	0.0070	0.1042	0.1967	0.0071	0.1031	0.1986	0.0072
Empirical std.dev.	0.0574	0.0603	0.0027	0.0298	0.0414	0.0015	0.0151	0.0244	2.0e-04
Empirical MSE	0.0035	0.0036	1.6e-05	0.0009	0.0017	1.1e-05	0.0002	0.0006	7.9e-06
$\theta_0 = (0.1, -0.2, 0.01)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1255	-0.1903	0.0072	0.1052	-0.1998	0.0071	0.0996	-0.2053	0.0071
Empirical std.dev.	0.1137	0.0624	4.0e-04	0.0250	0.0269	2.0e-04	0.0133	0.0169	1.0e-04
Empirical MSE	0.0136	0.0040	8.0e-06	0.0006	0.0007	8.4e-06	0.0002	0.0003	8.4e-06
$\theta_0 = (0.1, 0.3, 1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1017	0.2720	0.7647	0.1016	0.2862	0.7319	0.1012	0.2946	0.7191
Empirical std.dev.	0.0440	0.0676	0.0664	0.0253	0.0415	0.0371	0.0169	0.0228	0.0182
Empirical MSE	0.0019	0.0053	0.0598	0.0006	0.0019	0.0732	0.0003	0.0005	0.0792
$\theta_0 = (0.1, -0.3, 1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1118	-0.3045	0.7309	0.1055	-0.3074	0.7262	0.1004	-0.3105	0.7292
Empirical std.dev.	0.0537	0.0536	0.0272	0.0282	0.0272	0.0171	0.0171	0.0180	0.0104
Empirical MSE	0.0030	0.0029	0.0731	0.0008	0.0008	0.0752	0.0003	0.0004	0.0734
$\theta_0 = (0.2, 0.2, 0.05)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.2435	0.2182	0.0369	0.2021	0.1949	0.0357	0.2015	0.2003	0.0355
Empirical std.dev.	0.1634	0.1016	0.0038	0.0519	0.0466	0.0013	0.0324	0.0298	8.0e-04
Empirical MSE	0.0286	0.0107	0.0002	0.0027	0.0022	0.0002	0.0010	0.0009	0.0002
$\theta_0 = (0.2, -0.2, 0.05)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.2373	-0.1873	0.0362	0.2228	-0.193	0.0358	0.2047	-0.2006	0.0358
Empirical std.dev.	0.1397	0.0833	0.0021	0.1045	0.0564	0.0017	0.0345	0.0277	6.0e-04
Empirical MSE	0.0209	0.0071	0.0002	0.0114	0.0032	0.0002	0.0012	0.0008	0.0002

Table 1: Estimates of $\theta = (\alpha, \beta, \sigma)$ for different simulations. 100 samples generated.

	$T = 400$			$T = 1000$			$T = 2500$		
$\theta_0 = (0.5, 0.1, 0.5)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.5837	0.1153	0.3708	0.4885	0.0861	0.3525	0.4967	0.0959	0.3546
Empirical std.dev.	0.2843	0.1155	0.0485	0.1345	0.0690	0.0204	0.0899	0.0449	0.0140
Empirical MSE	0.0878	0.0136	0.0190	0.0182	0.0049	0.0222	0.0081	0.0020	0.0213
$\theta_0 = (0.5, -0.1, 0.5)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.5900	-0.0857	0.3691	0.5568	-0.0865	0.3618	0.5114	-0.0990	0.3553
Empirical std.dev.	0.3279	0.1164	0.0469	0.2127	0.0872	0.0293	0.1010	0.0461	0.0132
Empirical MSE	0.1156	0.0137	0.0193	0.0485	0.0078	0.0199	0.0103	0.0021	0.0211
$\theta_0 = (0.5, 0.4, 0.1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.5721	0.4021	0.0761	0.5110	0.3952	0.0716	0.5079	0.3994	0.0715
Empirical std.dev.	0.3066	0.1346	0.014	0.1502	0.0781	0.006	0.0922	0.0468	0.0036
Empirical MSE	0.0992	0.0181	0.0008	0.0227	0.0061	0.0008	0.0086	0.0022	0.0008
$\theta_0 = (0.5, -0.4, 0.1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.7586	-0.4459	0.0863	0.6150	-0.4353	0.0829	0.5675	-0.3957	0.0761
Empirical std.dev.	0.9176	0.1775	0.0265	0.6135	0.1415	0.0228	0.1772	0.0585	0.0079
Empirical MSE	0.9089	0.0336	0.0009	0.3896	0.0213	0.0008	0.0359	0.0034	0.0006
$\theta_0 = (0.8, 0.1, 0.01)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.8335	0.0874	0.0072	0.8264	0.0971	0.0072	0.8079	0.0993	0.0071
Empirical std.dev.	0.3621	0.1137	0.0011	0.1950	0.0696	5.0e-04	0.1147	0.0418	3.0e-04
Empirical MSE	0.1322	0.0131	9.0e-06	0.0387	0.0048	8.1e-06	0.0132	0.0017	8.5e-06
$\theta_0 = (0.8, -0.1, 0.01)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.8967	-0.0945	0.0074	0.8226	-0.1028	0.0071	0.8147	-0.1002	0.0071
Empirical std.dev.	0.4192	0.1166	0.0011	0.2392	0.0767	6.0e-04	0.1364	0.0464	3.0e-04
Empirical MSE	0.1851	0.0136	8.0e-06	0.0577	0.0059	8.8e-06	0.0188	0.0021	8.5e-06
$\theta_0 = (1, 0.2, 0.05)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	1.0403	0.1872	0.0348	0.9700	0.1822	0.0350	1.0299	0.2034	0.0357
Empirical std.dev.	0.3969	0.1171	0.0120	0.2279	0.0715	0.0031	0.1485	0.0431	0.0021
Empirical MSE	0.1591	0.0139	0.0004	0.0528	0.0054	0.0002	0.0229	0.0019	0.0002
$\theta_0 = (1, -0.2, 0.05)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	1.0292	-0.2253	0.0295	1.0048	-0.2114	0.0349	1.0342	-0.1991	0.0361
Empirical std.dev.	0.4708	0.1388	0.0215	0.3293	0.0972	0.0078	0.2282	0.0651	0.0027
Empirical MSE	0.2225	0.0199	0.0009	0.1085	0.0096	0.0003	0.0532	0.0042	0.0002

Table 2: Table 1 continued. Estimates of $\theta = (\alpha, \beta, \sigma)$ for different simulations. 100 samples generated.

	$T=400$			$T=1000$			$T=2500$		
$\theta_0=(0.1,0.1,0.1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1141	0.1015	0.0727	0.1033	0.0984	0.0715	0.1020	0.1006	0.0710
Empirical std.dev.	0.0879	0.0664	0.0052	0.0260	0.0350	0.0020	0.0161	0.0219	0.0012
Empirical MSE	0.0079	0.0044	0.0008	0.0007	0.0012	0.0008	0.0003	0.0005	0.0008
$\theta_0=(0.1,-0.1,0.1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1130	-0.0952	0.0715	0.1058	-0.0981	0.0711	0.1015	-0.1009	0.0708
Empirical std.dev.	0.0614	0.0532	0.0033	0.0285	0.0330	0.0017	0.0169	0.0210	0.0011
Empirical MSE	0.0039	0.0028	0.0008	0.0008	0.0011	0.0008	0.0003	0.0004	0.0008
$\theta_0=(0.1,0.2,0.01)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1120	0.1911	0.0069	0.1034	0.1966	0.0071	0.1028	0.1998	0.0071
Empirical std.dev.	0.0765	0.0664	0.0029	0.0254	0.0360	0.0014	0.0154	0.0215	7.0e-04
Empirical MSE	0.0060	0.0045	1.8e-05	0.0006	0.0013	1.0e-05	0.0002	0.0005	8.9e-06
$\theta_0=(0.1,-0.2,0.01)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1139	-0.1970	0.0071	0.1046	-0.2006	0.0071	0.1023	-0.2010	0.0071
Empirical std.dev.	0.0798	0.0547	0.0010	0.0262	0.0290	2.0e-04	0.0144	0.0178	1.0e-04
Empirical MSE	0.0066	0.0030	9.4e-06	0.0007	0.0008	8.4e-06	0.0002	0.0003	8.4e-06
$\theta_0=(0.1,0.3,1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1014	0.2671	0.7706	0.0977	0.2811	0.7333	0.0993	0.2927	0.7178
Empirical std.dev.	0.0728	0.0724	0.0903	0.0263	0.0425	0.0372	0.0163	0.0255	0.0187
Empirical MSE	0.0053	0.0063	0.0608	0.0007	0.0022	0.0725	0.0003	0.0007	0.0800
$\theta_0=(0.1,-0.3,1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1095	-0.3067	0.7283	0.0995	-0.3122	0.7268	0.1009	-0.3018	0.7110
Empirical std.dev.	0.0693	0.0512	0.0333	0.0266	0.0294	0.0176	0.0156	0.0157	0.0121
Empirical MSE	0.0049	0.0027	0.0749	0.0008	0.0010	0.0749	0.0002	0.00023	0.0837
$\theta_0=(0.2,0.2,0.05)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.2282	0.2032	0.0366	0.2123	0.2033	0.0358	0.2019	0.1987	0.0355
Empirical std.dev.	0.1448	0.0928	0.0034	0.0814	0.0562	0.0019	0.0339	0.0304	7.0e-04
Empirical MSE	0.0218	0.0086	0.0002	0.0068	0.0032	0.0002	0.0011	0.0009	0.0002
$\theta_0=(0.2,-0.2,0.05)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.2453	-0.1864	0.0362	0.2102	-0.1995	0.0358	0.2005	-0.2039	0.0356
Empirical std.dev.	0.1795	0.0909	0.0029	0.0717	0.0476	0.0012	0.0326	0.0263	6.0e-04
Empirical MSE	0.0343	0.0084	0.0002	0.0052	0.0023	0.0002	0.0011	0.0007	0.0002

Table 3: Estimates of $\theta=(\alpha,\beta,\sigma)$ for different simulations. 1000 samples generated.

	$T = 400$			$T = 1000$			$T = 2500$		
$\theta_0 = (0.5, 0.1, 0.5)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.5873	0.1143	0.3700	0.5335	0.1073	0.3595	0.5047	0.0984	0.3548
Empirical std.dev.	0.2987	0.1182	0.0499	0.1756	0.0788	0.0281	0.0966	0.0477	0.0146
Empirical MSE	0.0968	0.0142	0.0194	0.0319	0.0063	0.0205	0.0093	0.0023	0.0213
$\theta_0 = (0.5, -0.1, 0.5)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.5983	-0.0827	0.3682	0.5288	-0.0967	0.3584	0.5158	-0.097	0.3565
Empirical std.dev.	0.3175	0.1171	0.0456	0.1768	0.0775	0.0234	0.1118	0.0499	0.0144
Empirical MSE	0.1105	0.0140	0.0194	0.0321	0.0060	0.0206	0.0127	0.0025	0.0208
$\theta_0 = (0.5, 0.4, 0.1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.5763	0.4037	0.0756	0.5279	0.4014	0.0726	0.5094	0.4008	0.0714
Empirical std.dev.	0.3186	0.1389	0.0136	0.1754	0.0827	0.0072	0.0946	0.0481	0.0037
Empirical MSE	0.1073	0.0193	0.0008	0.0315	0.0068	0.0008	0.0090	0.0023	0.0008
$\theta_0 = (0.5, -0.4, 0.1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.6514	-0.4225	0.0836	0.5192	-0.4343	0.0799	0.5752	-0.3909	0.0767
Empirical std.dev.	0.5596	0.1677	0.0200	0.3265	0.1216	0.0110	0.1784	0.0575	0.0075
Empirical MSE	0.3361	0.0286	0.0007	0.1070	0.0160	0.0005	0.0375	0.0034	0.0006
$\theta_0 = (0.8, 0.1, 0.01)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.8682	0.0974	0.0073	0.8140	0.0936	0.0071	0.8060	0.0986	0.0071
Empirical std.dev.	0.3650	0.1134	0.0011	0.2076	0.0723	6.0e-04	0.1228	0.0440	3.0e-04
Empirical MSE	0.1379	0.0129	8.5e-06	0.0433	0.0053	8.8e-06	0.0151	0.0019	8.5e-06
$\theta_0 = (0.8, -0.1, 0.01)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.8640	-0.1025	0.0072	0.8197	-0.1021	0.0071	0.8076	-0.0864	0.0071
Empirical std.dev.	0.3934	0.1076	0.0011	0.2143	0.0698	5.0e-04	0.1305	0.0685	3.0e-04
Empirical MSE	0.1588	0.0116	9.0e-06	0.0463	0.0049	8.7e-06	0.0171	0.0049	8.5e-06
$\theta_0 = (1, 0.2, 0.05)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	1.0053	0.1737	0.0339	1.0152	0.1932	0.0355	1.0126	0.1999	0.0356
Empirical std.dev.	0.4250	0.1241	0.0128	0.2497	0.0723	0.0051	0.1412	0.0405	0.002
Empirical MSE	0.1806	0.0161	0.0004	0.0626	0.0053	0.0002	0.0201	0.0016	0.0002
$\theta_0 = (1, -0.2, 0.05)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	1.1304	-0.2002	0.0311	1.0389	-0.2009	0.0339	1.0327	-0.1942	0.0356
Empirical std.dev.	0.7469	0.1258	0.0209	0.3123	0.0803	0.0118	0.1973	0.0566	0.0043
Empirical MSE	0.5749	0.0158	0.0008	0.0990	0.0064	0.0004	0.0400	0.0032	0.0002

Table 4: Table 1 continued. Estimates of $\theta = (\alpha, \beta, \sigma)$ for different simulations. 1000 samples generated.

	$T = 150$			$T = 400$			$T = 1000$			$T = 2500$		
n = 100	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1564	0.0248	0.2937	0.1124	0.0110	0.2844	0.1028	0.0026	0.2833	0.1007	5.7e-07	0.2837
Empirical std.dev.	0.1615	0.1076	0.0355	0.0384	0.0522	0.0114	0.0254	0.0310	0.0071	0.0147	0.0190	0.0043
Empirical MSE	0.0293	0.0122	0.0125	0.0016	0.0028	0.0135	0.0006	0.0010	0.0137	0.0002	0.0004	0.0135
n = 1000	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	0.1690	0.0286	0.3012	0.1156	0.0045	0.2878	0.1047	-2.0e-04	0.2848	0.1021	1.0e-04	0.2834
Empirical std.dev.	0.2037	0.1238	0.0409	0.0666	0.0584	0.0146	0.0256	0.0334	0.0072	0.0155	0.0201	0.0044
Empirical MSE	0.0462	0.0161	0.0114	0.0047	0.0034	0.0128	0.0007	0.0011	0.0133	0.0002	0.0004	0.0136

Table 5: Estimates of $\theta = (0.1, 0, 0.4)$ for 100 and 1000 records samples.

	$T = 1000$			$T = 2500$			$T = 5000$		
$\theta_0 = (1.5, 0.3, 1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	1.4765	0.2857	0.7022	1.4796	0.2927	0.7048	1.4965	0.2981	0.7068
Empirical std.dev.	0.3407	0.0693	0.0863	0.1892	0.0345	0.0495	0.1251	0.0227	0.0319
Empirical MSE	0.1166	0.0050	0.0961	0.0362	0.0012	0.0896	0.0157	0.0005	0.0870
$\theta_0 = (3, 0.3, 1)$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
Empirical mean	3.1806	0.2951	0.7362	3.0025	0.2953	0.7098	3.0287	0.2991	0.7121
Empirical std.dev.	1.1813	0.0502	0.1991	0.4922	0.0283	0.0841	0.3222	0.0199	0.0535
Empirical MSE	1.4281	0.0025	0.1092	0.2423	0.0008	0.0913	0.1046	0.0004	0.0857

Table 6: 100 samples generated.

	$\ln(v(t))$		$v(t)$	
	procedure A	procedure B	procedure A	procedure B
Empirical mean	0.2951	0.2877	0.2834	0.2568
Empirical std.dev.	0.0502	0.0629	0.1063	0.0823
Empirical MSE	0.0025	0.0041	0.0116	0.0086

Table 7: Estimation β with procedure A and procedure B for $\theta_0 = (3, 0.3, 1)$. over sets of 400 records.

	$T = 1000$		$T = 2500$	
	$\ln(v(t))$	$v(t)$	$\ln(v(t))$	$v(t)$
Empirical mean	0.2951	0.2669	0.2953	0.2693
Empirical std.dev.	0.0502	0.0588	0.0283	0.0394
Empirical MSE	0.0025	0.0045	0.0008	0.0025

Table 8: Estimates of β with procedure A for processes $\ln(v(t))$ and $v(t)$.

T	20	40	60	80	100	400	1000
procedure A	0.84	1.63	2.33	3.46	4.16	19.65	77.32
procedure B ($h = \frac{1}{20}$)	3.37	12.29	28.24	51.88	83.6	1755.34	13283.93
procedure B ($h = \frac{1}{100}$)	16.75	61.78	141.59	260.64	423.22	8813.54	66147.45

Table 9: Comparison of user CPU time (in seconds) between data simulation with procedure A and procedure B.

T	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
500	1.810	-0.0979	0.0195
2000	2.4566	-0.0444	0.0188
10000	1.8824	0.0053	0.0131
16128	2.4165	0.0120	0.0197

Table 10: Estimation of S&P 500 Stock Index parameters.

data	lag1	2	5	10	20	40	70	100
$T = 500$								
W_t	0.0734	-0.0458	0.0250	0.0559	-0.0320	-0.0255	0.0047	0.0215
$\lvert W_t\rvert^{1/2}$	-0.0004	0.1165	0.1307	0.0844	0.0605	-0.0128	0.0430	-0.0052
$\lvert W_t\rvert$	0.0325	0.1671	0.1575	0.1293	0.092	-0.0141	0.0061	-0.0004
$\lvert W_t\rvert^2$	0.0784	0.2433	0.1699	0.1573	0.1117	-0.0094	-0.0283	0.0225
$T = 2000$								
W_t	0.0494	-0.0057	-0.0090	0.0142	0.0012	-0.0209	0.0263	0.0177
$\lvert W_t\rvert^{1/2}$	-0.0214	-0.0072	0.0826	0.0222	0.0280	-0.0040	0.0359	0.0001
$\lvert W_t\rvert$	-0.0029	0.0187	0.0997	0.0258	0.0505	0.0036	0.0422	-0.0020
$\lvert W_t\rvert^2$	0.0401	0.0562	0.1153	0.0275	0.0668	0.0018	0.0376	-0.0045
$T = 10000$								
W_t	0.1580	-0.0224	0.0122	0.0125	0.0036	0.0079	0.0028	0.0071
$\lvert W_t\rvert^{1/2}$	0.1161	0.0813	0.1196	0.0867	0.0789	0.0601	0.0775	0.0550
$\lvert W_t\rvert$	0.1223	0.0986	0.1326	0.0989	0.0944	0.0702	0.0879	0.0622
$\lvert W_t\rvert^2$	0.1065	0.1044	0.1281	0.0937	0.0988	0.0698	0.0847	0.0559
$T = 16127$								
W_t	0.0971	-0.0362	0.0054	0.0180	0.0036	0.0222	-0.0061	0.0041
$\lvert W_t\rvert^{1/2}$	0.1783	0.1674	0.1879	0.1581	0.1567	0.1371	0.1252	0.1293
$\lvert W_t\rvert$	0.2044	0.2012	0.2215	0.1831	0.1835	0.1596	0.1439	0.1464
$\lvert W_t\rvert^2$	0.1864	0.2018	0.2220	0.1684	0.1709	0.1510	0.1303	0.1321

Table 11: Autocorrelation of W_t, $|W|^d$ for $d = 1/2, 1, 2$ for the S&P 500.

Transformation	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}$
V_t (truncated by the 1% and 99% quantiles)	0.5322	-0.1440	0.4846
V_t (truncated by the 2% and 98% quantiles)	0.2336	-0.3199	0.3737
W_t (without truncation)	0.7019	-0.0359	0.0774
W_t (truncated by the 1% and 99% quantiles)	0.7091	-0.0342	0.0648

Table 12: Estimation of the T-Bill rate Parameters.

data	lag1	2	5	10	20	40	70	100
V_t	0.4014	0.1789	0.0285	0.1389	0.1759	0.1377	0.0419	0.0193
$\|V_t\|^{1/2}$	0.4244	0.3385	0.3012	0.2279	0.2377	0.2010	0.1882	-0.0626
$\|V_t\|$	0.3999	0.3593	0.2698	0.2622	0.2125	0.1696	0.1636	-0.0691
$\|V_t\|^2$	0.2423	0.2551	0.1626	0.2476	0.1089	0.0890	0.0899	-0.0570
W_t	0.446	0.1961	0.003	0.1212	0.128	0.1306	0.062	-0.0598
$\|W_t\|^{1/2}$	0.2893	0.1589	0.1781	0.1028	0.1381	0.1143	0.1174	0.0165
$\|W_t\|$	0.2803	0.1654	0.1654	0.1525	0.1306	0.1235	0.1250	0.0278
$\|W_t\|^2$	0.2108	0.1395	0.1020	0.1861	0.1146	0.1102	0.1314	0.0722

Table 13: Autocorrelation of V_t, $|V_t|^d$, W_t and $|W_t|^d$ for $d = 1/1, 1, 2$ for the T–Bill rate.