"Extreme Distribution of Realized and Range-based Risk Measures"[♣]

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Preliminary draft

Abstract

The study of market volatility has been renewed by the introduction of high-frequency estimators. The popularity of the so-called realized volatility as a proxy of latent, unobservable volatility has been facilitated by the increased access to transaction data. Yet the recent studies do not totally agree on the empirical properties of that variable, especially its exact distribution.

While Andersen *et al* (2001) or Cizeau *et al* (2001) fit a log-normal to the volatility series, Liu *et al* (1999) find that the distribution is right-side asymmetric and heavy tailed and Thomakos and Wang (2003) report a low power of moment-based normality tests. In the current study, we apply several measures of risk on high-frequency data and test the goodness-of-it of possible candidate distributions for the volatility. Of particular interest is the estimation of the right-hand tail of the distribution, since it represents the periods of crisis, when the market is at its most volatile.

In order to include important shocks and crises, it is best to have a sample spanning a long period, where high-frequency data is not available. Thus we extend the study by using range-based estimators (*Cf.* Brandt and Diebold, 2003, and among others, Li and Weinbaum, 2000, Lebaron, 2001, Corrado and Miller, 2002, Alizadeh *et al*, 2002, Tims and Mahieu, 2003) and comparing them, whenever possible with high-frequency estimators.

Combining different risk measures (from classical variance to squared maximum drawdown) and distributional assumptions (from log-normal to inverse Gamma), we retrieve the likelihood of benchmarks market events such as the successive crises of the last 10 years (with high frequency data) and the famous historical crises from 1929 to 1987 (with daily data). We use extreme value theory to characterize the slope of the right hand tail of the volatility distribution and get characteristic return times for the market shocks. The result allows us to rank historical crises and to check whether the worst of them can fit in the general process or are in fact outliers as suggested for instance by Sornette *et al* (2003).

Keywords: Financial Crisis, Realized Volatility, Extreme Value and Range-based Volatility Estimators, High Frequency Data.

J.E.L. Classification: G.10, G.14.

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1. Introduction

Following for instance Barndorff-Nielsen and Shephard (2003) or Andersen *et al* (2003), volatility can be viewed as a latent factor (the so-called quadratic variation affecting the Brownian motion in some representations) that can only be estimated using its signature on market prices. It is only when the process is known (and simulated) as in Andersen and Bollerslev (1997, 1998) or Barndorff-Nielsen and Shephard (2002-a) - that we know what the "True Volatility" is (inside the Brownian motion case - see Andersen *et al*, 2003, for a survey). The results are not so clear when the underlying process is more sophisticated as shown by Barndorff-Nielsen and Shephard (2002-b) or when observed prices suffer from market microstructure distortions effects (see Corsi *et al*, 2001, Andersen, *et al*, 2000, Bai *et al,* 2001, Oomen, 2002, Brandt and Diebold, 2003).

Realized volatility has been, since its introduction, considered the best estimator for the latent factor. The daily volatility retrieved from transaction data has been shown to be accurate when controlling for microstructure effect and empirically supporting the mixture of distribution hypothesis (Andersen *et al*, 2000). Among the high-frequency estimators, the one using all the available transactions (VARHAC estimator, Bollen and Inder, 2002) has been shown to perform better than the realized volatilities that use a lower sampling rate.

When high-frequency data is unavailable, second best estimations of the unobservable risk factor are provided by the range-based – or extreme value – estimators. The price range, defined as the difference between the highest and lowest market prices over a fixed sampling interval, is known for a long time as a proxy for volatility estimator. Several authors[1], back to Parkinson (1980), develop extreme value estimators of volatility far more efficient than the classical empirical variance – using various assumptions on the underlying process.

[1] Relevant literature includes Parkinson (1980), Garman and Klass (1980), Rogers and Satchell (1991), Kunitomo (1992) and Yang and Zhang (2000). For the sake of exhaustivity, we should also mentioned works by Ball and Torous (1984), Brunetti and Lildholdt (2002), Bollen and Inder (2002, 2003) and, Høg and Lunde (2003).

It is now generally admitted since the seminal papers by Cizeau *et al* (1997) and Andersen *et al* (2001 and 2001) that the log-volatility is approximately Gaussian for a daily integrated-horizon. Nevertheless, the hypothesis is challenged by different facts. The first one comes from the following intuition: on one extreme, it is well know that the unconditional ultra high-frequency return density is leptokurtic, due to the presence of fat tails but also to a pick around zero-return (no-event); the density of the volatility at these frequencies should be then degenerated with a high level of probability around zero (zero-return, zero-volatility); in the other extreme, unconditional low frequency return density is proven to be approximately Gaussian and then the density of the volatility at a low frequency should then be closed to a Khi-squared like-shaped law with some degrees of freedom. In other words, as underlined by Forsberg and Bollerslev (2002), the time-aggregation effect should also be found for this feature of a financial variable. The second fact yielding questions is the amplitude of some extreme events, specifically 1929 and 1987 crises that might be a challenging issue when attempting to attached a probability to such huge (squared) variations. The third fact is based on Miccichè *et al* (2002) finding that the right tail of a log-normal density largely underestimate the empirical (kernel) estimation of volatility density, and Forsberg and Bollerslev (2002) that show that the Normal Inverse Gaussian is a reasonable distribution for the variance - accounting for main induced scale-relation parameters - in the Mixing Distribution Hypothesis of Clark (1973).

In the first part of this article, we will compute various estimates of volatility on a sample of French stocks, using both transaction data and range-based estimators.
The second part will be devoted to the study of the distributional properties of these estimators, testing the goodness of fit of several candidate probability distributions.
Finally, in the third part, we will fit extreme value distributions to the right-hand tail of daily volatilities in order to get estimated frequencies and return times of extreme events.

2. Realized and Range-based Daily Measures of Risk

2.1 Symmetric Measures of Risk and Extreme Value Estimators of Volatility

In several financial applications, return dispersions are estimated by calculating the standard deviation of a time series of daily, weekly, or monthly returns based on a security's closing prices.

Let us first consider ***p*** time-periods τ (corresponding to a frequency of interest which will be hereafter the daily frequency for the sake of simplicity), each containing price observations dated by time ***t*** and labelled by ***i***, ***i*** varying from 1 to I_p, where I_p is the (random) number of prices within the period. The instantaneous driftless high-frequency empirical standard deviation computed at time ***t*** (end of the window of the ***p*** time-periods τ) and at a time-scale τ (end of the window of the period ***p***), traditionally reads[2]:

$$\sigma_t = \left[\eta \sum_{i=1}^{I_p} \left[\ln\left(P_{t_i} / P_{t_i + \Delta_\tau} \right) \right]^2 \right]^{\frac{1}{2}}$$

where $\eta = n_b / \tau$ is a factor[3] for annualizing the time-scale volatility, n_b is the number of business days ***per*** year, τ is the time-resolution expressed in number of observations ***per*** day, $\{P_{t_i}\}$, $i = [1,...,I]$, is the sequence of prices of the asset at time t_i with $t_i = t - i\tau/I = [1,...,I]$, the date of observation of return involved in the time-scale τ volatility and $\Delta_\tau = \tau/I$ is the time-increment depending on the time scale.

When it comes to low-frequency series (daily frequency for instance), a mean adjusted variant of the mean empirical standard deviation, traditionally used as a proxy for the volatility, becomes:

$$\hat{\sigma}_t = \left[\frac{1}{(N-1)} \sum_{n=t-N}^{t} \left[\ln(P_n / P_{n-1}) - \hat{\mu}_t \right]^2 \right]^{\frac{1}{2}}$$

where N is estimation window expressed in number of business days, dates ***t*** correspond to ends of business days, $\{P_n\}$ is a sequence of closing prices and:

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[2] Note that we do not applied the correction mentioned by Figelwski (1998) to unbiaised the standard deviation estimation since it is proven to be very small in sample (see Poon and Granger, 2003).

[3] Note that, by contrast to Martens (2002), Areal and Taylor (2002) and Hol and Koopman (2002), we do not correct here the volatility for the presence of noisy overnight returns since we do not focus on a special time-scale accurate measure, but rather on a specific time-serie observation scale corresponding to economic agent observation frequency.

$$\hat{\mu}_t = \frac{1}{N} \sum_{n=t-N}^{t} \left[\ln\left(P_n / P_{n-1}\right) \right]$$

is an estimation of the mean log-return on the reference period. s pointed by Poon and Granger (2002), since the statistical properties of sample mean make it a very inaccurate estimate of the true mean especially for small sample, taking deviations around zero - or around a very long period mean - instead of the sample mean increases volatility estimate accuracy.

This classical standard deviation of return series is assumed to be a proxy for the future dispersion of returns. This is easy to compute and not information demanding but, when using only closing prices - that corresponds to quite arbitrary calendar dates, some information is ignored concerning the path of the price inside the period of reference. Instead of using the closing prices, Parkinson (1980) propose to use the information provide by the high/low records to improve the estimator of the volatility. His estimator then reads (with previous notation):

$$\hat{\sigma}^P{}_t = \left[\frac{1}{\theta_N} \sum_{n=1}^{N} \left[\ln\left(H_n / L_n\right) \right]^2 \right]^{\frac{1}{2}}$$

where:

$$\begin{cases} H_n = \underset{t_i}{Arg}\left[\underset{P_{t_i}}{Max}\left\{ P_{t_i} \mid t_i \in [n-1, n] \right\} \right] \text{is the highest price at day n} \\ L_n = \underset{t_i}{Arg}\left[\underset{P_{t_i}}{Min}\left\{ P_{t_i} \mid t_i \in [n-1, n] \right\} \right] \text{is the lowest price at day n} \\ \theta_N = 4N \ln(2) \text{ is a correction parameter} \end{cases}$$

The Parkinson (1980) extreme value estimator efficiency intuitively comes from the fact that the range of intradaily quotes gives more information regarding the future volatility than two arbitrary points in this series (the close prices). Note nevertheless that we need to know all the sequence of intra-daily quotes to be able to compute this extreme value volatility estimator, which this time is quite information demanding[4]. When having access to this high/low records, we gain information from high-frequency data and make a great improvement when used in financial applications. Roughly speaking, knowing these records allows us to get closer to the "real underlying process", even if we do not know the whole path of asset prices.

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[4] **Nevertheless, most financial data providers give access to High and Low price series.**

But using four data points - open, close, high and low prices - instead of two - close-to-close or high-low prices - can also give extra information - especially if they are followed by all market participants and are references for some part of the exchanges. Garman and Klass (1980) propose an estimator based on the knowledge of the open, close, high and low prices that can be written (with previous notation):

$$\hat{\sigma}^{GK}{}_t = \left[\alpha_1 \sum_{n=t-N}^{t} \left[\ln\left(H_n / L_n \right) \right]^2 + \alpha_2 \sum_{n=t-N}^{t} \left[\ln\left(C_n / C_{n-1} \right) \right]^2 \right]^{\frac{1}{2}}$$

where C_n **is the closing price at day n,** α_1 **and** α_2 **are weighting parameters such as:** $\alpha_1 = .5 / N$ **and** $\alpha_2 = .39 / N$.

Since Parkinson (1980) and Garman and Klass (1980) estimators implicitly assume that log-stock prices follow a geometric Brownian motion with no drift, further refinements are given by Rogers and Satchell (1991) and Kunitomo (1992). The latter author uses the open and close prices to estimate a modified range corresponding to a hypothesis of a Brownian bridge of the transformed log-price. This basically aims to correct the high and low prices for the presence of a drift (with previous notation):

$$\hat{\sigma}^{K}{}_t = \left[\frac{1}{\beta_N} \sum_{p=t-N}^{t} \left[\ln\left(\hat{H}_n / \hat{L}_n \right) \right]^2 \right]^{\frac{1}{2}}$$

where:

$$\begin{cases} \hat{H}_n = \underset{t_i}{Arg} \left[\underset{P_{t_i}}{Max} \left\{ P_{t_i} - \left[O_n + \left(C_n - O_n \right) / t_i \right] + \left(C_n - O_n \right) \middle| \, t_i \in [n-1, n] \right\} \right] \\ \qquad \textit{is the end - of - the - day projection of drift - corrrected highest price at day n} \\ \hat{L}_n = \underset{t_i}{Arg} \left[\underset{P_{t_i}}{Min} \left\{ P_{t_i} - \left[O_n + \left(C_n - O_n \right) / t_i \right] + \left(C_n - O_n \right) \middle| \, t_i \in [n-1, n] \right\} \right] \\ \qquad \textit{is the end - of - the - day drift - corrrected lowest price at day n} \\ \beta_N = 6 / \left(N \pi^2 \right) \textit{ is a correction parameter} \end{cases}$$

Rogers and Satchell (1991) also add a drift term in the stochastic process that can be incorporated into a volatility estimator using only daily open, high, low, and closing prices, that reads (with previous notation):

$$\hat{\sigma}^{RS}{}_{t} = \left[\frac{1}{N} \sum_{n=t-N}^{t} [\ln(H_n/O_n)[\ln(H_n/O_n) - \ln(C_n/O_n)] + \ln(L_n/O_n)[\ln(L_n/O_n) - \ln(C_n/O_n)]] \right]^{\frac{1}{2}}$$

where O_n is the open price at day *n*.

They also propose an adjustment that is designed to take into account the fact that one may not be able to continuously monitor the stock price. Their adjusted estimator is the positive root of the following quadratic equation:

$$\hat{\sigma}^{ARS}{}_{t} := \underset{\hat{\sigma}^{ARS}{}_{t} > 0}{Arg} \left[\frac{.5594}{I_n} \hat{\sigma}^{ARS}{}_{t}{}^{2} + \frac{.9072}{\sqrt{I_n}} \ln(H_n/L_n) \hat{\sigma}^{ARS}{}_{t} + \hat{\sigma}^{RS}{}_{t}{}^{2} = 0 \right]$$

with I_n is the total number of transactions occurring during day *n*.

Finally, Yang and Zhang (2000) make further refinements by deriving an extreme-value estimator that is unbiased, independent of any drift, and consistent in the presence of opening price jumps. Their estimator thus writes (with previous notation):

$$\hat{\sigma}^{YZ}{}_{t} = \left[\frac{1}{(N-1)} \sum_{n=t-N}^{t} \left[\ln(O_n/C_{n-1}) - \overline{\ln(O_n/C_{n-1})} \right]^2 + \frac{\kappa}{(N-1)} \sum_{p=t-N}^{t} \left[\ln(C_n/O_n) - \overline{\ln(C_n/O_n)} \right]^2 + (1-\kappa) \hat{\sigma}^{RS}{}_{t}{}^{2} \right]^{\frac{1}{2}}$$

with:

$$\kappa = \frac{.34}{\left[1.34 + \frac{N+1}{(N-1)} \right]}$$

where $\bar{x}$ being the unconditional mean of *x* and $\hat{\sigma}^{RS}{}_{t}{}^{2}$ being the Rogers-Satchell estimator (see above text).

The Yang-Zhang estimator is simply the square root of the sum of the estimated overnight variance (the first term on the right hand side) and the estimated open market variance (which is a weighted average of the open market return sample variance) and the Rogers and Satchell (1991) drift independent estimator (where the weights are chosen so as to minimize the variance of the estimator). The resulting estimator therefore explicitly incorporates a term for the closed market variance (that is the overnight variance).

As crucially emphasized Alizadeh *et al* (2002), range-based estimators have many attractive properties over either low frequency estimators or even, for some authors, high-frequency based volatility estimators.

The range is a highly efficient volatility proxy as shown by Brandt and Diebold (2003) in a multivariate setting. First, it is a potent distillation of the valuable information about volatility contained in the entire intraday price path, whereas alternative volatility proxies based on the daily return use only (the opening and) closing prices. In other words, one might experience very turbulent days with long swing, drop and recovery of the markets where traditional close-to-close volatility indicates a low regime despite this large intraday price fluctuation. The daily range, in contrast, reflects the intraday price fluctuations and therefore indicates correctly that the volatility is high. Second, the range appears robust to common forms of market microstructure noise, such as bid-ask bounce. Indeed, as pointed by Brandt and Diebold (2003), the intuition is simple: the observed daily maximum is general reach at the ask and hence is too high by half the spread, whilst the observed minimum is likely to be attained at the bid and therefore is too low by half the spread. Generally speaking, the bid-ask bounce inflates the range by the average spread, which is small in general in liquid markets and quiet market conditions. Third, even the price range is inefficient when compared to high-frequency estimators (see Andersen and Bollerslev, 1998, p. 898, footnote 20, and Andersen *et al*, 2001) in the Brownian motion case, it is far from clear that range-based estimators are less efficient outside the Brownian case or in the presence of market microstructure effects, simply because the problem is difficult to handle theoretically and because microstructure noise recovers a large collection of potential biases which are difficult either to properly defined or explained (Crack and Ledoit, 1996).

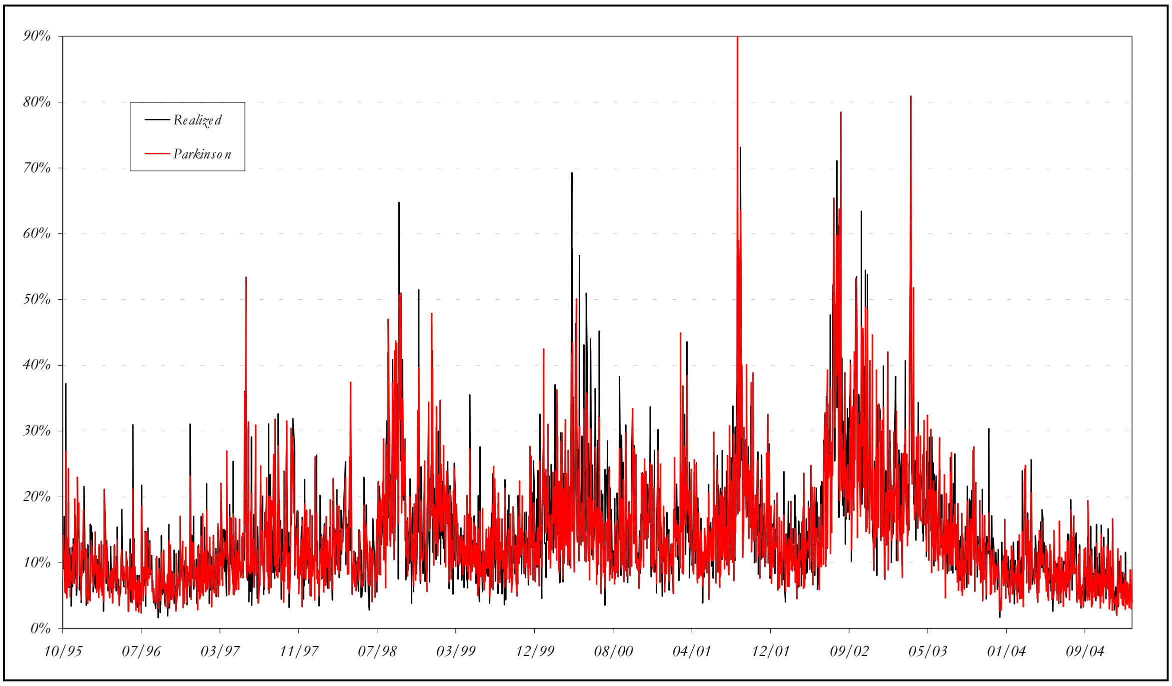
In term of efficiency (measured in term of the ratio of variance of the extreme value estimator over the classical one), all previous estimators exhibit very substantial improvements. Parkinson (1980) reports a theoretical relative efficiency gain ranging from 2.5 to 5, while the Garman and Klass (1980), Yang and Zhang (2000) and Kunitomo (1992) variance estimators result in a theoretical efficiency gain of, respectively, 7.4, 7.3, and 10. Rogers *et al* (1994) report that the Rogers-Satchell estimator yields theoretical efficiency gains comparable to the Garman-Klass estimator. They also report that the Rogers-Satchell estimator appears to perform well with changing drift and as few as 30 daily observations.

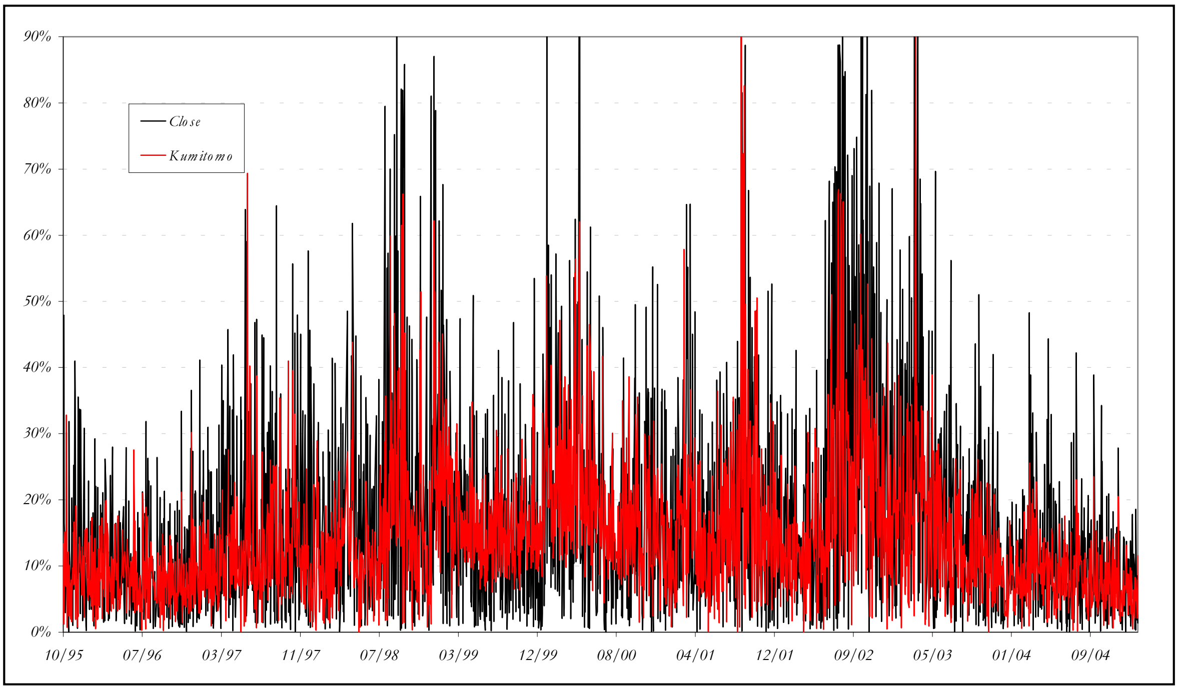
Consistent with the previous research of Wiggins (1991), Li and Weibaum (2002) find that the Parkinson estimator - applied to daily S&P 100 and S&P 500 data covering periods of 1 to 24 (trading) days over the period January 1989 - December 1999 - is downward biased compared to the traditional estimator, at the weekly and monthly frequencies. This is also true for the other extreme value volatility estimators. They provide reasonable reasons to explain this result and other coherent extreme value estimates on S&P 500 Futures and exchange rates markets (see Li and Weibaum, 2000, p.19 and following). Corrado and Miller (2002) nevertheless report that empirical estimates of extreme value volatilities exhibit high cross-correlation coefficients (from .908 to .999; see Corrado and Miller, 2002, Table 1, p.26) on the S&P 100 and S&P 500 American indexes based on monthly volatility intervals from January 1988 through December 1999. They also show that they are all highly correlated with the Implied Standard Deviation (correlation between .715 to .769; see Corrado and Miller, 2002, Table 1, p.26) backed-out from the American option market using the traditional Black-Scholes' formula.

The set of Figures 1 represents the various weekly estimates - namely Realized Volatility, Classical Empirical Variance, Parkinson (1980), Garman and Klass (1980), Rogers and Satchell (1991), Kunitomo (1992), Yang and Zhang (2000) Extreme Value Variance estimations - of daily volatility using intraday quotes of the 30'' sampled CAC40 French stock index quotes on the period 01-1991/07-2002. These numbers are square root of variance estimates, annualized by multiplying it by the square root of the number of trading days *per* annum divided by the number of days in its volatility interval (see, Hull, 2000).

The peaks of the variance estimates are approximately synchronous, but the general behaviour of the series differs, both in the range of variances and persistence phenomenon (see section 3).

Figures 1: Weekly Estimates of Volatilities

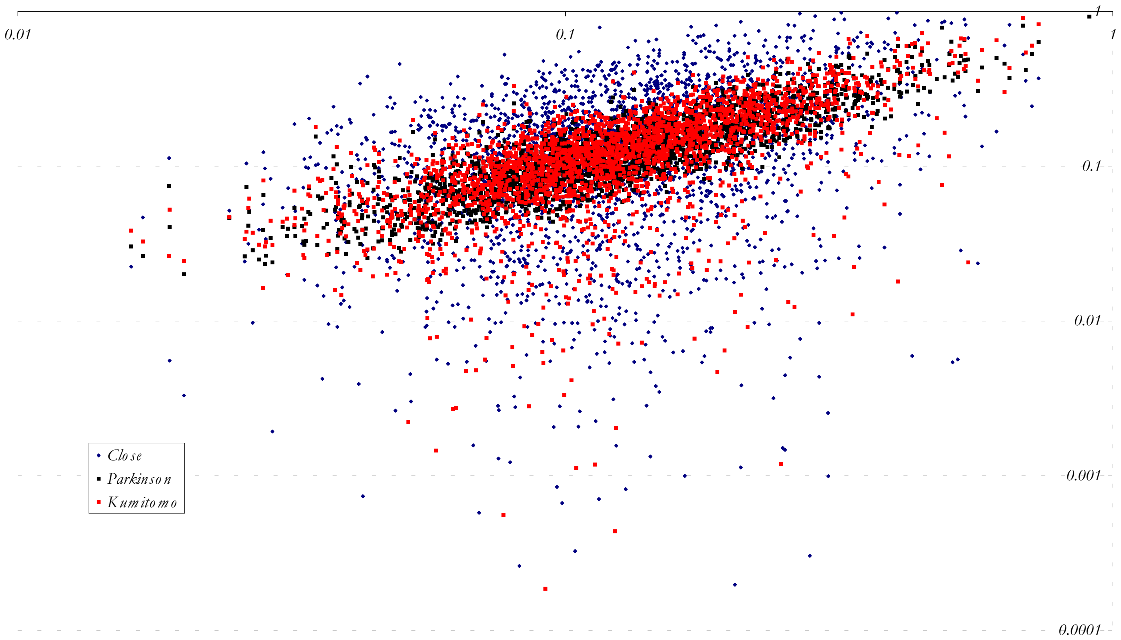




Source: Euronext, 30" sampled intraday CAC40 French stock index quotes on the period 01-1991/07-2002. Computations by the authors. Realized Volatility, Classical Empirical Variance, Parkinson (1980), Kunitomo (1992) Extreme Value Variance estimations are presented here as examples.

The following graph represents the main measures of volatility plotted against the realized volatility (used as benchmark).

Figures 2: Weekly Estimates of Volatilities against Realized Volatility



Source: Euronext, 30" sampled intraday CAC40 French stock index quotes on the period 01-1991/07-2002. Computations by the authors. Sorted Realized Volatility ont the x-axis versus *Classical Empirical Variance, Parkinson (1980) and Kunitomo (1992) Extreme Value Variance estimations on the y-axis.*

2.2 Alternative Asymmetric Measures of Risk

When returns are skewed and leptokurtic, better measures of volatility have been discussed in the literature. For instance, semi-variance estimator only focuses on the left part of the return distribution that corresponds to true losses for investors. The rescaled[5] mean semi-standard deviation reads (with previous notations):

$$\hat{\sigma}^{SV}{}_t = \left[\frac{2}{(N-1)} \sum_{n=t-N}^{t} \left[Min\left[\ln(P_n / P_{n-1}) - \hat{\mu}_t , 0 \right] \right]^2 \right]^{\frac{1}{2}}$$

while its DownSide Risk mean empirical counterpart equals to:

[5] We multiply the semi-variance (and DownSide Risk Measure) by two since if the underlying density is symmetric, the semi-variance is equal to half the variance. Taking the square root of this rescaled estimator give then a number that is of the same order that standard deviation and mean of returns.

$$\hat{\sigma}^{DSR}{}_t = \left[\frac{2}{(N-1)} \sum_{n=t-N}^{t} \left[Min\left[\ln\left(P_n / P_{n-1}\right), 0 \right] \right]^2 \right]^{\frac{1}{2}}$$

This adaptation is particularly efficient when extreme variations are asymmetric. Using asymmetric measure of risk better characterizes the drop in the market and do not signal periods of booms where semi-volatility is low.[6] Note furthermore that high/low ratio used in the Parkinson's estimator (see text above) is close to a classical risk measure called drawdown; if we substitute the drawdown to the high/low ratio into the Parkinson's formula we get a mean squared drawdown measure[7] of risk that reads (with previous notation):

$$\hat{\sigma}^{mDD}{}_t = \left[\frac{1}{\gamma_N} \sum_{n=t-N}^{t} \left[\ln\left(H_n / L^*{}_n \right) \right]^2 \right]^{\frac{1}{2}}$$

where:

$$\begin{cases} L^*{}_n = \underset{t_j}{Arg} \left[\underset{P_{t_j}}{Min} \left\{ P_{t_j} \mid t_j > t_i, (t_i, t_j) \in [n-1, n]^2 \right\} \right] \text{ is the lowest price following the highest price at day n} \\ \gamma_N \text{ is a scaling parameter} \end{cases}$$

By using the drawdown instead of the Parkinson's measure of volatility, we also focus on negative returns that are under interest for assessing crises. Drawdown quantifies the financial losses in a conservative way: it adds losses for the most "unfavorable" investment moments - buy at the highest, sell at the lowest - in the reference period. This approach reflects quite well the preferences of investors who define their allowed losses in percentages of their initial investments. While an investor may excuse short-term drawdowns in his account, he would definitely start worrying about his capital in the case of a long-lasting drawdown. Such a drawdown may indicate that something is wrong with a specific market and maybe it is time to retrieve the money and place it in a more successful investment pool (an other markets: gold,

6 Note, however, that the higest period volatility generally correspond to a first large decrease in prices followed by a second recovering in prices. In other words, the largest moves in the markets are drops in price, and not booms (see Goodhart and Danielsson, 2002 and Maillet and Michel, 2003-b).

7 Note that this definition differs from the α-CDaR (mentioned in Krokhal *et al*, 2002) in the sense that first it is the unconditional mean of recent drawdowns (on the window estimation) and second it does not depend upon a (conditional) threshold; note also that we slightly modify the traditional definition of drawdown (see Johanssen and Sornette, 2001) since they correspond here to the worst investment timings within the estimation window. That allows to avoid the difficult problem of *ex ante* determining the highest and lowest price on the reference period (see Bouchaud et al for mentioning this problem of determining local extrema).

property funds, mutual fund in CHF, bond markets, bear funds...). Thus, one can conclude that drawdown accounts not only for the amount of losses, but also for the duration of these losses. As pointed by Krokhal *et al* (2002) and roughly speaking, this highlights a feature of the drawdown concept: it is a loss measure "with memory" taking into account the magnitude of losses but also the time sequence of losses. That also is an interesting characteristic for our purpose in defining an IMS on a sound risk measure, since a long range of small negative returns will entail a high drawdown, characterizing a decreasing trend in prices with a low volatility[8].

For stressing the extreme variation of prices, we can also substitute - being furthermore conservative - the maximum to the average drawdown in the previous formula that now reads:

$$\hat{\sigma}^{MDD}{}_t = \left[\frac{1}{\lambda_N} \left[\ln\left(H^{**}{}_{t_i} / L^{**}{}_{t_j} \right) \right]^2 \right]^{\frac{1}{2}}$$

with:

$$\begin{cases} H^{**}{}_{t_i} = \underset{t_i}{Arg}\left[\underset{P_{t_i}}{Max}\left\{ P_{t_i} \mid t_i \in [t-N,t] \right\} \right] \text{ is the highest price on the period } [t-N,t] \\ L^{**}{}_{t_j} = \underset{t_j}{Arg}\left[\underset{P_{t_j}}{Min}\left\{ P_{t_j} \mid t_j \in [t_i,t]^2 \right\} \right] \text{ is the lowest price following the highest price on the period } [t-N,t] \\ \lambda_N \text{ is rescaling parameter} \end{cases}$$

An illustration of such measures on the French market in provided in Figure 3.

3. A Global Statistical Comparison of Symmetric and Asymmetric Measures of Risk

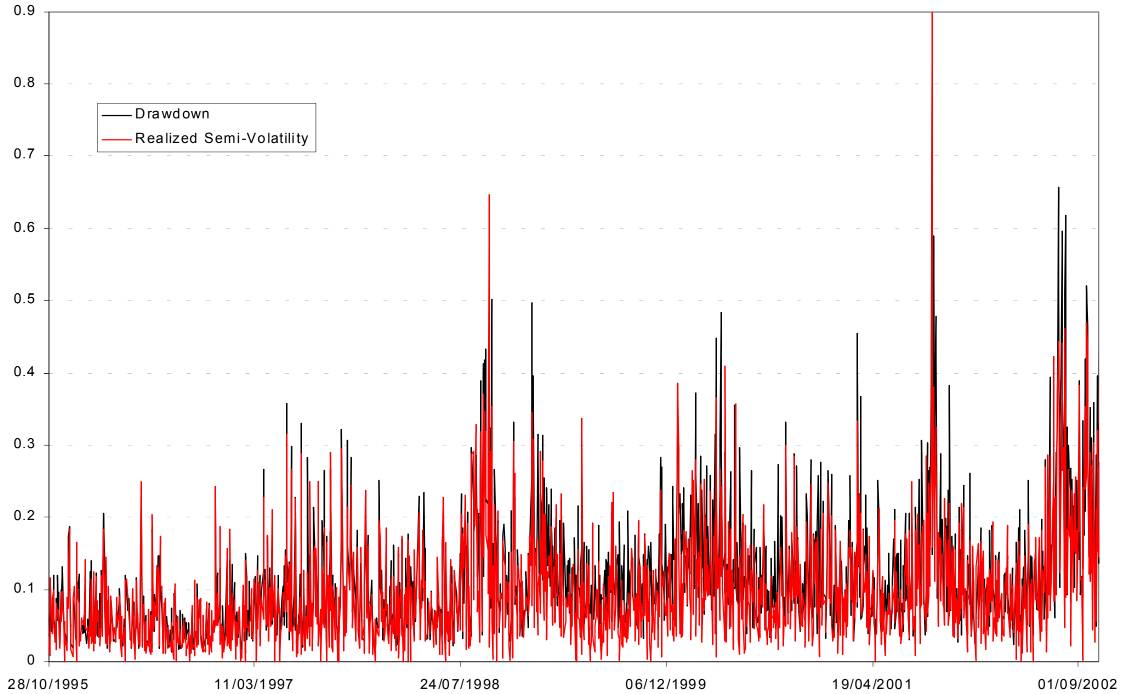
If the general definition of all measures is similar, the analysis of both statistical properties of the measures of risk time series sheds light on the difference between each measure. We present some statistical features of all series as well as correlations and related distributional characteristics.

[8] As experienced at the end of May and beginning of June 02 where stock markets - measured by a world index for instance - get down regularly at a small rate on a daily basis. The volatility - measured on a short window - was low during this period since dispersion of returns around the mean was also low. Using the drawdown measure instead of the classical allows to make the risk measure more adequate to was is intuitively a crisis.

3.1 Descriptive Statistics and Correlations

Table 1 presents the main characteristics of risk measures from the moments to the autocorrelation coefficients.

Figures 3: Weekly Estimates of Asymmetric Risk Measures



Source: Euronext, 30'' sampled intraday CAC40 French stock index quotes on the period 01-1991/07-2002. Computations by the authors. Drawdonw and Realized Semi-volatility are presented in this Figure.

Table 1: Statistics of the Log-Volatilities

Volatility	Skewness	Kurtosis	Jarque-Bera	Kolomogorov-Smirnov	1 % critical
Realized	-0.03	3.43	0.05%	0.0182	0.0366
Close	-1.14	5.28	0.00%	0.0822	0.0366
Parkinson	0.21	3.04	0.06%	0.0218	0.0366
Kumitomo	-1.22	6.81	0.00%	0.0726	0.0366
Drawdown	-0.09	2.87	13.98%	0.0143	0.0366

As already seen in Figure 1, estimators using intra-day data are less volatile (more accurate) than the classical estimator. Series are both asymmetric and fat tailed and thus the Gaussian hypothesis for log-estimations is generally rejected. All measures exhibit high linear persistence as seen from the first partial autocorrelation.

The following table corresponds to the correlation matrixes of risk log-estimations.

Table 2: Pearson Correlations between Risk Measures

	Realized	Close	Parkinson	Kumitomo	Drawdown
Realized	1.00	0.28	0.83	0.54	0.67
Close	0.28	1.00	0.36	0.06	0.16
Parkinso	0.83	0.36	1.00	0.54	0.70
Kumitomo	0.54	0.06	0.54	1.00	0.55
Drawdown	0.67	0.16	0.70	0.55	1.00

Parkinson's volatility is very close to the realized volatility and will be used as a proxy when all the data is not available intra-daily. The other intraday measure (Kunitomo) is also highly correlated with these two, in contrast with Garman-Klass and Rogers-Satchell estimators. Lastly the squared close-to-close squared return evolution is relatively poorly correlated with all other measures. These results are confirmed by the analysis of the relative rankings of the observed market sessions according to their volatility (Spearman correlation between measures).

Table 3: Spearman Rank Correlations between Risk Measures

	Realized	Close	Parkinson	Garman-Klass	Kumitomo	Rogers-Satchell	Drawdown	Realized Semi-vol
Realized	1.00	0.33	0.82	0.49	0.64	0.20	0.66	0.71
Close	0.33	1.00	0.40	-0.33	0.14	0.11	0.19	0.16
Parkinson	0.82	0.40	1.00	0.58	0.69	0.24	0.71	0.53
Garman-Klass	0.49	-0.33	0.58	1.00	0.53	0.12	0.49	0.36
Kumitomo	0.64	0.14	0.69	0.53	1.00	0.16	0.65	0.47
Rogers-Satchell	0.20	0.11	0.24	0.12	0.16	1.00	0.53	0.41
Drawdown	0.66	0.19	0.71	0.49	0.65	0.53	1.00	0.80
Realized Semi-vol	0.71	0.16	0.53	0.36	0.47	0.41	0.80	1.00

3.2 Distributional Properties

Quantity of articles have been dedicated to distribution of financial returns (see for instance Mc Donald, 1994), but relatively few on the distribution of empirical volatility (see, for instance, Andersen *et al*, 2001 or Thomakos and Wang, 2003). Distributional properties of risk measures are important features for our purpose of measuring the financial crisis. Several papers advocate that log-volatility are normally distributed (see Andersen *et al*, 2001) while others show that Beta, Pearson type V or stretched exponential densities exhibit better fits depending on the measure, the data and the time-scale. Indeed, as it has been shown for raw returns, one might expect the shape of the volatility distribution to be deformed when increasing the time scale

resolution. For instance, it is well known that ultra high-frequency data (corresponding to quotes-to-quotes intra-day for a stock for instance) exhibit a very sharp peak around zero due to the parsimony of information arrival at this time scale. One might think that cause a spurious zero mode in the volatility density. When cumulating price variations (decreasing the observation frequency), zero movements in volatility become less frequent, and the mode should displace to a positive number while the probability of observing volatility equal to zero should be lower.

Natural candidates for representing risk estimation distribution are the following:

- **the log-normal (see, for instance, Cizeau *et al*, 1997 or Andersen *et al*, 2001):**

$$F_{\ln}(\sigma) = \frac{1}{\hat{\sigma}_\sigma \sqrt{2\pi}} \int_{s=0}^{\sigma} \exp\left\{ \frac{-\left[\ln(s) - \hat{\mu}_\sigma\right]^2}{2\hat{\sigma}_\sigma^{\ 2}} \right\} ds$$

with $\hat{\mu}_\sigma$ and $\hat{\sigma}_\sigma$ being the empirical mean and standard deviation of the volatility.

- **the more general and flexible Beta distribution (See Johnson and Kotz, 1995 and Kotz, 2002:**

$$F_B(\sigma) = \int_{s=0}^{\sigma} \frac{s^\alpha (1-s)^\beta}{B(\alpha,\beta)} ds$$

with the scaling Beta function such as: $B(\alpha,\beta) = \int_{\sigma=0}^{1} \sigma^\alpha (1-\sigma)^\beta d\sigma$ **for** $(\alpha,\beta) \in IR^{+2}$**.**

- **the Pearson type V (or inverted Gamma, see Miccichè *et al*, 2002):**

$$F_{PV}(\sigma) = 1 - F_B(1/\sigma)$$

where $F_B(.)$ is the Beta distribution.

The Hull-White hypothesis (see Hull and White, 1987):

$$F_{HW}(\sigma) = 2\frac{\left(ba/\xi^2\right)^{1+a/\xi^2}}{\Gamma\left(1+a/\xi^2\right)} \int_{s=0}^{\sigma} \frac{\left[\exp\left(-ba/\xi^2\right)s^2\right]}{s^{2a/\xi^2+3}} ds$$

- **the transformed Hull-White hypothesis (see Miccichè *et al*, 2002):**

$$F_{MBLM}(\sigma) = \frac{\left(ba/\xi^2\right)^{1+a/\xi^2}}{\Gamma\left(1+a/\xi^2\right)} \int_{s=0}^{\sigma} \frac{\left[\exp\left(-ba/\xi^2 s\right)\right]}{s^{a/\xi^2+2}} ds$$

where the corresponding probability density function is the inverted gamma:

$$f(\sigma) = \frac{B^C}{\Gamma(C)} \cdot \frac{\exp(-\frac{B}{\sigma})}{\sigma^{C+1}}$$

where $B = \frac{ba}{\xi^2}$ and $C = 1 + \frac{a}{\xi^2}$.

- the Stretched Exponential (Johanssen and Sornette, 2001):

$$F_{SE}(\sigma) = \frac{1}{\beta}\left[\exp\left(-\frac{\sigma}{\beta}\right)\right]^{\alpha}$$

with the scale parameter being $\beta = 1/b^{1/\alpha}$, ***b*** a constant and $\alpha > 1$.

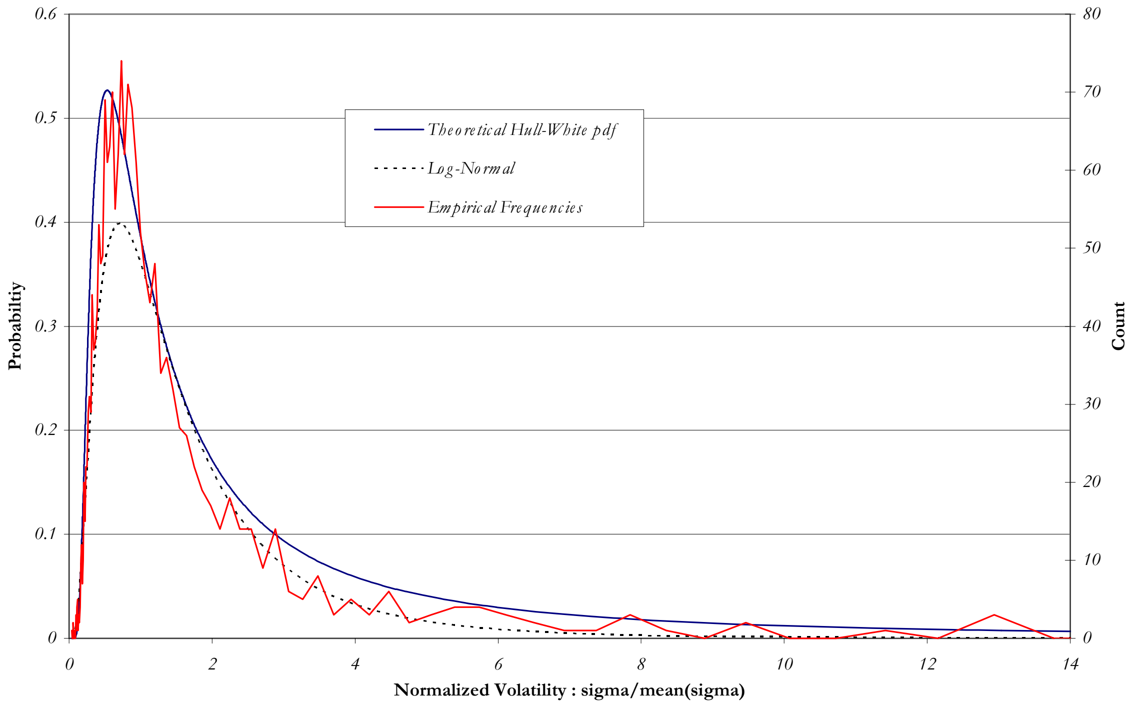
- The Normal Inverse Gaussian (Barndorff-Nielsen, 1997) :

(to be completed)

All scale and shape parameters are estimated using matching moment maximum likelihood estimation method (see, for instance, Law and Kelton, 1991, for these conditions) and cumulative density functions performed using numerical approximations.

As an example, the next figure illustrates different hypothesis for the distribution of the 30' realized volatility.

Figures 4: Empirical Estimation of Volatility Density Function



Source: Euronext, 30'' sampled intraday CAC40 French stock index quotes on the period 01-1991/07-2002. Computations by the authors.

The log-normal seems to underestimate the right tail of the volatility (under interest here) whilst the inverted gamma has a tendency to overestimates the extreme volatility values.
Table 4 presents the results of goodness-of-fit tests for all measures, for all the sampling frequencies and all distributions considered (to be completed).

- Please Insert Table 4 somewhere here -

4. Extreme Values of the Daily Risk Estimates

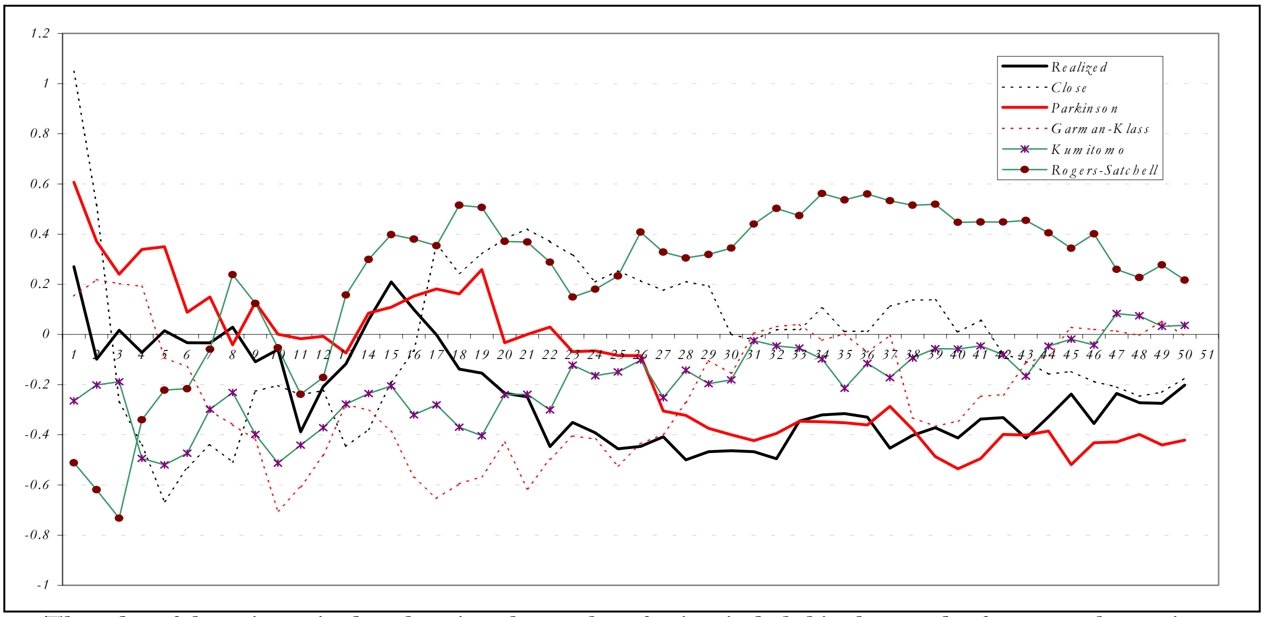
According to the central result of extreme value theory, the extrema of the measures of the risk should converge asymptotically to the GEV distribution. This distribution is characterized by the parameter known as the tail index, measuring the rate of decreases of the probability in the tails.

$$H_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-\frac{1}{\xi}}) \; if \; \xi \neq 0 \\ \exp(-e^{-x}) \; if \; \xi = 0 \end{cases}$$

Three basins of attraction can be distinguished according to the value of the tail index ξ: negative, zero or positive. In the case of fat-tailed distributions, the tail index will be significantly positive, whereas the gaussian distribution yields ξ=0 (the Gumbel distribution).

The GEV distribution needs to be estimated on the maxima of the random variable, which comprise necessarily a relatively small sample. Several methods, parametric and non-parametric can be used to retrieve the value of the tail index, and for the sake of robustness we present here the different estimates.

Figure 5: Tail Index estimated by the Non-Parametric Method of Pickands



The value of the estimate is plotted against the number of points included in the sample of extreme observations.

Table 5: Tail Indexes of Weekly Maxima of Log-Volatility via L-moment Method

Realized	Close	Parkinson	Garman-Klass	Kumitomo
-0.046	-0.119	-0.048	-0.078	-0.053

An alternative method is to use not the sample of the maxima, but peak-over-threshold, that is values of the random variable that exceed a cut-off point. In that case, the asymptotic distribution is the generalized Pareto distribution (GPD) which can also be estimated by parametric or non-parametric methods. Table 6 presents the results of the estimation of the GPD for various measures and methods.

- *Please insert Table 6 somewhere here –*

Given the sample estimates of the parameters, it is now possible to compute the probability of observing the historical volatility peaks under the various measures and hypotheses. Table 7 presents these probabilities and allows us to exclude the combinations of measures and distributional assumptions that are unlikely given the sample size.

- *Please insert Table 7 somewhere here –*

5. Conclusion

Using the realized volatility as a benchmark and based also on its distributional properties, we select the Parkinson volatility as the best range-based estimate in terms of accuracy and availability. The log-normal approximation is shown to be a reasonable approximation, and the asymptotic distribution of the extrema of the log-volatility doesn't seem to correspond with an underlying heavy-tailed distribution.

A practical application of these results will be to plug the appropriate estimates and distributions in the Index of Market Shocks (IMS, see Maillet and Michel, 2003 and 2004) in order to get an more accurate ranking of the historical crises and a precise estimation of the return times of extreme *scenarii*.

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